Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 3. Deadline: October 28th, 2003

Exercise 8 (Relative constructibility) *[Repeated from Set # 2]*.
Let x be a real number (pick your favourite set-theoretic concept of "real number": Dedekind cuts, Cauchy sequences, subsets of ω, *etc.*). Show that $\mathbf{L}(x) = \mathbf{L}[x]$. Furthermore, argue why this result doesn't depend on the choice of the concept of "real number" you chose.

Exercise 9 (The Continuum and $\aleph_2$).
Prove using relative constructibility that if $\mathsf{ZFC} \vdash 2^{\aleph_0} \neq \aleph_2$, then $\mathsf{ZFC} \vdash \mathsf{CH}$. (Of course, Cohen has proved in 1963 that the antecedent of this implication is false, so the exercise has a trivial solution. Do not use this fact. You may use the result that for $A \subseteq \kappa^+$, we have that $\mathbf{L}[A] \models 2^\kappa = \kappa^+$.)
Hint. Assume $2^{\aleph_0} > \aleph_1$. Find an appropriate $A \subseteq \aleph_2$ such that $\mathbf{L}[A] \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

Exercise 10 (Measurable cardinals and relativized constructibility).
Let κ be the least measurable cardinal and $A \subseteq \kappa$. Show that $\mathbf{V} \neq \mathbf{L}(A)$.

Exercise 11 (More on strong cardinals).
Remember from **Exercise 6** that a cardinal κ is called **$\kappa{+}\alpha$-strong** if there is an elementary embedding $j : \mathbf{V} \prec M$ with critical point κ such that $\mathbf{V}_{\kappa+\alpha} \subseteq M$. We call a cardinal **strong** if it is α-strong for all ordinals α. (Note that the embeddings witnessing α-strength can be different from each other.)
Formulate and prove extensions of **Exercise 10** for α-strong and strong cardinals.

Exercise 12 (Weakly compact cardinals and linear orderings).
Let κ be weakly compact, *i.e.*, $\kappa \to (\kappa)^2_2$. Let $A \subseteq \kappa$ and let $\preceq$ be a linear ordering on A. Show that there is a subset $B \subseteq A$ of cardinality κ such that either $\langle B, \preceq\rangle$ or $\langle B, \succeq\rangle$ is wellordered.

Exercise 13 (Infinitary partition relations and ultrafilters).
Let $\lambda < \kappa$ be regular, and assume that for all $\gamma < \kappa$, we have $\kappa \to (\kappa)^\lambda_\gamma$. (Note that this assumption contradicts the Axiom of Choice, so be sure to work in ZF in this exercise.)
Let

$$\mathcal{C}^\lambda_\kappa := \{X \subseteq \kappa\,;\, \exists C \in \mathcal{C}_\kappa\,(C \cap \{\xi\,;\,\mathrm{cf}(\xi) = \lambda\} \subseteq X\}$$

the so-called **λ-club filter on κ**. Show that under the assumption of the above partition relation, it is a κ-complete ultrafilter.
Hint. If $\langle X_\alpha\,;\,\alpha < \gamma\rangle$ is a partition of κ, let H be a homogeneous set for the colouring $\chi : [\kappa]^\lambda \to \gamma$ with

$$\chi(S) = \alpha \iff \sup S \in X_\alpha.$$

Show that $C := \{\xi\,;\,\sup(H \cap \xi) = \xi \,\&\, \mathrm{cf}(\xi) = \lambda\}$ witnesses that some $X_\alpha \in \mathcal{C}^\lambda_\kappa$.