Universiteit van Amsterdam
Institute for Logic, Language and Computation

Advanced Topics in Set Theory

2003/2004; 1st Semester
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Homework Set # 1 (Warm-up exercises). Deadline: September 16th, 2003

Exercise 1

Let κ be an inaccessible cardinal. In the lecture, we checked that $\mathbf{V}_\kappa \models (\mathsf{Ers})$. Check that all of the other axioms of ZFC hold in $\mathbf{V}_\kappa$.

Exercise 2

Let $(\infty\mathsf{IC})$ be the axiom saying that for each $\alpha \in \mathrm{Ord}$, there is an inaccessible cardinal $\kappa > \alpha$.
As in the lecture, call κ **hyperinaccessible** if κ is inaccessible, and the set of inaccessible cardinals below κ has cardinality κ.

(1) Show that $(\infty\mathsf{IC})$ doesn't imply the existence of a hyperinaccessible cardinal.
(2) Show that the least hyperinaccessible cardinal κ_0 is not a Mahlo cardinal.
Hint. Use the function $F(\alpha) \mapsto$ the αth inaccessible cardinal, and look at $F^{-1}\restriction\kappa_0$ (which is a regressive function on the set of inaccessible cardinals below κ_0).

Remark. This was Exercise 6.3 in last year's *Axiomatic Set Theory* class. Unfortunately, the students there were asked to prove the (false) negation of claim (1).

Exercise 3

Recall that a (class) function $F : \mathrm{Ord} \to \mathrm{Ord}$ is called **normal** if it is increasing $(\forall\alpha < \beta\,(F(\alpha) < F(\beta)))$ and continuous (for all limit λ, $F(\lambda) = \bigcup\{F(\alpha)\,;\,\alpha < \lambda\}$). Also recall that it is provable in ZFC that every normal function has a fixed point. (If you have never seen a proof of this, prove it.)
Call the statement "every normal function has a regular fixed point" the **regular fixed point axiom** RFPA.

(1) Show that RFPA is not provable in ZFC.
Hint. Think of a normal function (normally denoted by a Hebrew letter) whose regular fixed points would be inaccessibles.
(2) Show that if κ is Mahlo, then $\mathbf{V}_\kappa \models \mathsf{RFPA}$.