# Toric varieties II

**Exercise 1.**
Let $\Sigma, \Sigma'$ be fans in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$, respectively, with toric varieties $X(\Sigma), X(\Sigma')$. Let $\varphi\colon \Sigma \to \Sigma'$ be a morphism of fans. Let $\Phi\colon X(\Sigma) \to X(\Sigma')$ be the induced morphism.

i. $\Phi$ is a closed immersion if and only if for all $\sigma' \in \Sigma'$ there is a $\sigma \in \Sigma$ with $\varphi^{-1}(\sigma') = \sigma$ such that $(\sigma')^\vee \cap M' \to \sigma^\vee \cap M$ is surjective.
ii. $\Phi$ is dominant if and only if $N'/N$ is a torsion group.

**Exercise 2.**
Let $\Sigma, \Sigma'$ be fans in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$, respectively. Let $\varphi\colon N \to N'$ be a surjective $\mathbb{Z}$-linear morphism inducing a morphism of fans $\varphi\colon \Sigma \to \Sigma'$. Consider the exact sequence

$$0 \to N_0 \to N \to N' \to 0.$$

Define $\Sigma_0 = \{\sigma \in \Sigma \mid \sigma \subset (N_0)_{\mathbb{R}}\}$.

i. Check that $\Sigma_0$ is a fan in $(N_0)_{\mathbb{R}}$.
ii. Let $\sigma, \sigma'$ be cones in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ respectively, such that $\varphi$ maps $\sigma$ bijectively onto $\sigma'$. Prove that $\varphi\colon N \to N'$ has a splitting $\mu$ such that $\mu$ maps $\sigma'$ to $\sigma$.

**Exercise 3.** In the situation of exercise 2, the fan $\Sigma$ is *split* by $\Sigma'$ and $\Sigma_0$ if there is a subfan $\hat{\Sigma} \subset \Sigma$ such that :

i. $\varphi$ maps each cone $\hat{\sigma} \in \hat{\Sigma}$ bijectively to a cone $\sigma' \in \Sigma'$ such that $\varphi(\hat{\sigma} \cap N) = \sigma' \cap N'$. Moreover, $\hat{\sigma} \to \sigma'$ is a bijection $\hat{\Sigma} \to \Sigma'$.
ii. Given cones $\hat{\sigma} \in \hat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, the sum $\hat{\sigma} + \sigma_0$ lies in $\Sigma$, and every cone of $\Sigma$ arises this way.

Show that if $\Sigma$ is split, then $X(\Sigma')$ has a cover by affine open subsets $U$ such that

$$\Phi^{-1}(U) \cong X(\Sigma_0) \times U.$$

Deduce that all fibres of $X(\Sigma) \to X(\Sigma')$ are isomorphic to $X(\Sigma_0)$.

**Exercise 4.** Let $r$ be a nonnegative integer. The fan $\Sigma_r$in $N_{\mathbb{R}} = \mathbb{R}^2$ given by the four cones $\mathrm{cone}(-e_1+re_2, e_2), \mathrm{cone}(-e_1+re_2, -e_2), \mathrm{cone}(e_2, e_1), \mathrm{cone}(e_1, -e_2)$ defines a toric variety, the Hirzebruch surface $\mathcal{H}_r$.

i. Show that $\mathcal{H}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Show that there is no toric isomorphism $\mathcal{H}_r \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $r > 0$.
ii. Check that the projection on the first factor $\pi_1\colon \mathbb{Z}^2 \to \mathbb{Z}$ determines a toric morphism $\mathcal{H}_r \to \mathbb{P}^1$.

iii. Find a fan $\Sigma_0$ such that $\Sigma_r$ is split by the fan of $\mathbb{P}^1$ and $\Sigma_0$. Deduce that $\mathcal{H}_r$ is a locally trivial fibration over $\mathbb{P}^1$.