**Algebraic geometry II** Prof. Dr. Bernd Siebert
Winter 2015/16 Carsten Liese

# Toric varieties I

**Exercise** 1.
Let $d$ be a positive integer and $e_1, e_2$ the standard basis of $\mathbb{R}^2$. Show that the toric variety $U_\sigma$ determined by the cone $\sigma = (de_1 - e_2, e_2)$ is isomorphic to $\mathrm{Spec}[X_0, \ldots, X_d]/I$ where $I$ is the ideal generated by the $2 \times 2$ minors of

$$\begin{pmatrix} X_0 & \ldots X_{d-1} \\ X_1 & \ldots X_d \end{pmatrix}$$

This variety is known as the rational normal cone.

**Exercise** 2.
Classify all the 1-dimensional toric varieties.

**Exercise** 3.

i. Let $\sigma$ be a cone in $N$ and $\sigma'$ a cone in $N'$. Show that $\sigma \times \sigma'$ is a cone in $N \oplus N'$ and construct an isomorphism
$$U_{\sigma \times \sigma'} \cong U_\sigma \times U'_\sigma$$
.
ii. Let $\Sigma, \Sigma'$ be fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$. Show:
$$\Sigma \times \Sigma' := \{\sigma \times \sigma' \ |\sigma \in \Sigma, \sigma' in \Sigma'\}$$
is a fan in $N_\mathbb{R} \times N'_\mathbb{R}$. Also, show
$$X(\Sigma \times \Sigma') \cong X(\Sigma) \times X(\Sigma').$$

**Exercise** 4. Let $S = C \cap M$, $A = k[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$. Suppose $A = k[M]$ with $M = \mathbb{Z}S$. The action $\alpha$ of the torus $\mathbb{G}_m^n = \mathrm{Spec}(A)$ on the affine toric variety $U = \mathrm{Spec}(k[S])$ is given by a morphism
$$\alpha : \mathbb{G}_m^n \times U \to U$$
satisfying a number of axioms. These can also be stated in terms of the coaction
$$\rho : k[S] \to k[S] \otimes A$$
of $\alpha$. We require
$$(id_A \otimes \epsilon) \circ \rho = id_A$$
and
$$(id_A \otimes \triangle) \circ \rho = (\rho \otimes id_A) \circ \rho.$$

Here, $\triangle$ is the comultiplication given by $\triangle = \mu^* \otimes \cdots \otimes \mu^*$, where

$$\mu^* : k[T, T^{-1}] \to k[T, T^{-1}] \otimes k[T, T^{-1}]$$

is the comultiplication on the factors of $\mathbb{G}_m^n$ and is given by $\mu^*(T) = T \otimes T$. Similarly, $\epsilon$ is the product $e^* \otimes \cdots \otimes e^*$ of the counits $e^* : k[T, T^{-1}] \to k$ given by $e^*(T) = 1$.

Check that the algebra homomorphism $k[S] \to k[M] \otimes k[S]$ given by $\chi^m \to \chi^m \otimes \chi^m$ is a coaction. This is the algebra homomorphism defining the torus action.