Universität Hamburg
Department Mathematik

Center for Mathematical Physics Hamburg

Workshop on Holonomy Groups and Applications in String Theory, Hamburg July 14 - 18, 2008

Speakers, Titles, and Abstracts

Homogeneous para-Kähler Einstein manifolds

Dmitri V Alekseevsky, University of Edinburgh

A para-Kähler manifold can be defined as a pseudo-Riemannian manifold (M, g) with a parallel skew-symmetric para-complex structures K, i.e. a parallel field of skew-symmetric endomorphisms with $K^2 = \mathrm{Id}$ or, equivalently, as a symplectic manifold (M, ω) with a bi-Lagrangian structure $L^{\pm}$, i.e. two complementary integrable Lagrangian distributions.
A homogeneous manifold $M = G/H$ of a semisimple Lie group G admits an invariant para-Kähler structure (g, K) if and only if it is a covering of the adjoint orbit $\mathrm{Ad}_G h$ of a semisimple element h. We give a description of all invariant para-Kähler structures (g, K) on a such homogeneous manifold. Using a para-complex analogue of basic formulas of Kähler geometry, we prove that any invariant para-complex structure K on $M = G/H$ defines a unique para-Kähler Einstein structure (g, K) with a given non-zero scalar curvature. An explicit formula for the Einstein metric g is given.
The talk is based on a joint work with C.Medori and A.Tomassini (Parma University).

Parallel spinors and the spinorial Weierstrass representation in 3+1 dimensions

Bernd Ammann, Universität Regensburg

Let N be a compact $(n+1)$-dimensional Riemannian manifold carrying a parallel spinor ψ and let M be a hypersurface. Then the parallel spinor on N restricts to a spinor ϕ on M. This restricted spinor has constant length and solves

$$D\phi = H\phi$$

where D is the Dirac operator and where H is the mean curvature of M in N.

We consider the converse: Assume that a spinor ϕ on M has constant length and $D\phi = H\phi$. Is there (at least locally) an ambient manifold N carrying a parallel spinor ψ that extends the spinor ϕ?
The answer is yes for $n = 2$, but in high dimensions the answer is expected to be negative. This problem was solved for $n = 2$ and $H = 0$ already by Weierstrass in 1866. The Weierstrass representation expresses minimal surfaces in R^3 locally in terms of two holomorpic functions on the surface, and this data can be interpreted as a solution of $D\phi = 0$. Obviously neither the language of spinors was available at that time nor were global aspects considered. The complete 2-dimensional case was solved by Abresch, Kusner, Schmitt and others between 1990 and 1995.
In our talk we consider the case $n = 3$. We show the following: Assume we have a real-analytic solution of $D\phi = H\phi$ on a real-analytic 3-dimensional Riemannian manifold M. Then there is a 4-dimensional manifold N carrying a parallel spinor such that M is a hypersurface of N with mean curvature H. Thus N is a manifold with holonmy $SU(2)$ or equivalently a Calabi-Yau manifold, i.e. a Kähler manifold with vanishing Ricci curvature. We can construct counteresamples showing that the real analycity is necessary.
The proof uses a Riemannian analogue of the Einstein constraint equations from General Relativity.

Holonomy Groups from 1924 to 1955

Marcel Berger, IHÉS

Motivated by General relativity Élie Cartan introduced in 1924 the notion of holonomy group for an affine connection. Motivated by the geometry of Riemannian manifolds, from 1926 to 1927, he used heavily the notion of holonomy group for inventing the notion of symmetric spaces, and completely study and classified them.
Then holonomy groups were essentially completely forgotten, until Lichnerowicz, with the help of Armand Borel, Chevalley, Nijenhuis, Ambrose-Singer and de Rham, in 1952-1953, got basic results for them, namely compactness and reductibility. Plus the fact that Kähler manifolds are exactly those whose holonomy group is contained in the unitary group.
But a priori any compact subgroup of the orthogonal group could be the holonomy group of a suitable Riemannian manifold. But in 1953-1955 it was discovered that, except for the very rigid symmetric spaces, only five series of holonomy groups could exist, plus very few exceptions in low dimensions.
The talk will give a historical description of these events from 1924 to 1935.

Calabi-Yau Metrics and the Spectrum of the Laplace Operator

Volker Braun, University of Pennsilvania

Finding the spectrum of the Laplacian is important for the Kaluza-Klein compactification of string theory down to 4 dimensions. Starting with Donaldson's algorithm to numerically compute the Calabi-Yau metric, I will present numerical solutions to the Laplace-Beltrami operator. As a simple application, one can compute threshold corrections to Newton's law. Finally, I will give some numerical evidence for the SYZ conjecture.

Harmonic almost contact structures via the intrinsic torsion

Francisco Martín Cabrera, Universidad de La Laguna

We go further on the study of harmonicity for almost contact metric structures already initiated by Vergara and Wood. By using the intrinsic torsion, we characterise harmonic almost contact metric structures in several equivalent ways and show conditions relating harmonicity and classes of almost contact metric structures. Additionally, the harmonicity as a map of almost contact metric structures is considered. Finally, it is shown that the standard a-Sasakian structure defined on odd-dimensional round spheres gives the absolute minimum for the energy. (Joint work with José Carmelo González-Dávila)

Genus 1 Lefschetz fibrations for generalized complex structures

Gil Cavalcanti, Oxford University

I'll talk about work in progress with Gualtieri where we study different genus 1 Lefschetz fibrations which arise in generalized complex geometry. I'll also give examples and counter-examples for natural questions about such fibrations.

Special geometry with solvable Lie groups

Simon Chiossi, Politecnico di Torino

I will try to persuade the audience that nilpotent and solvable Lie groups are a fitting playground for so-called special geometry.

Complex Submanifolds and Holonomy

Sergio Console, Università di Torino

After presenting some general properties of the holonomy of the normal connection of (complex) submanifolds of (complex) space forms, I will describe the computation of the holonomy group $\Phi^\perp$ of the normal connection of complex parallel submanifolds of $\mathbb{C}P^n$. A by-product is a new proof of the classification of complex parallel submanifolds of $\mathbb{C}P^n$ by using a normal holonomy approach (joint work with A. J. Di Scala [1]).
Next I will present some Berger type theorems for $\Phi^\perp$ (joint work with C. Olmos and A. J. Di Scala [2]). Namely,

1. for $\mathbb{C}^n$, if M is irreducible, then $\Phi^\perp$ acts transitively on the unit sphere of the normal space;
2. for $\mathbb{C}P^n$, if $\Phi^\perp$ does not act transitively, then M is the complex orbit, in the complex projective space, of the isotropy representation of an irreducible Hermitian symmetric space of rank greater or equal to 3.

[1] S. Console, A. Di Scala, Parallel submanifolds of complex projective space and their normal holonomy, *Math. Z.* (DOI 10.1007/s00209-008-0307-8) (electronic)

[2] S. Console, A.J. Di Scala, C. Olmos, A Berger type normal holonomy theorem for complex submanifolds, preprint (2008).

The symplectic duality of Hermitian symmetric spaces

Antonio J. Di Scala, Politecnico di Torino

In this talk we describe the *symplectic duality map* $\Psi : M^n \to \mathbb{C}^n$ of an Hermitian symmetric space M of non-compact type. This map was introduced in [DL]. The main property of Ψ is to be a bi-symplectorphism, namely, $\Psi^*\omega_0 = \omega_{hyp}$ and $\Psi^*\omega_{FS} = \omega_0$, where ω_0 is the flat symplectic form of M (i.e. the restriction of ω_0 to M regarded as a bounded domain of $\mathbb{C}^n$ in its circled realization), ω_{hyp} is the hyperbolic form on M (i.e. ω_{hyp} is the Kähler form of the symmetric space M that is equal to ω_0 at the tangent space T_0M of the origen $0 \in M$) and ω_{FS} is the Fubini-Study form on the affine chart $\mathbb{C}^n \subset \mathbb{C}P^n$. Then we will discuss the unicity problem of such a map Ψ, i.e. to what extent this map is unique. This last part is based on the work [DLR]. (Joint work with Andrea Loi and Guy Roos)

[DL] Di Scala, A.J. and Loi, A., *Symplectic Duality of Symmetric Spaces,* Advances in Mathematics 217 (2008) 2336-2352.

[DLR] Di Scala, A.J. ; Loi, A. and Roos, G. *The unicity of the symplectic duality,* To appear in Transformation Groups.

AdS/CFT correspondence and differential geometry

Johanna Erdmenger, Max-Planck-Institut München

The AdS/CFT correspondence (AdS: Anti-de Sitter space, CFT: Conformal field theory) provides a relation between quantum field theory and classical general relativity which is based on string theory. In particular, in its original form it relates field-theoretical properties of $N = 4$ super Yang-Mills theory, such as correlation functions and the conformal anomaly, to geometrical properties of the space $AdS_5 x S^5$. Recently, the correspondence has been generalised to quantum field theories in which supersymmetry is partially broken. In this case the five-sphere S^5 is replaced by manifolds with a more involved metric. An example are quiver gauge theories which are related to Sasaki-Einstein manifolds. In the talk an outline of these results is given and further studies of the generalised AdS/CFT correspondence are presented.

Lie n-algebras and supersymmetry

José M. Figueroa-O'Farrill, University of Edinburgh

A Lie n-algebra is a ternary generalisation of a Lie algebra, complementary (in a sense) to a Lie triple system. After reviewing their emergence in supersymmetric theories, and motivating the recent interest in these algebras, I will present some recent classification results on metric Lie n-algebras.

Contact manifolds and $SU(n)$-structures

Anna Fino, Università di Torino

Any oriented hypersurface of a Calabi-Yau manifold of real dimension 6 is equipped with a *hypo structure* in the sense of Conti and Salamon. In the first part of the talk, I will consider 5-manifolds with a *hypo-contact* structure, i.e. with a contact form arising from a hypo structure. I will show a classification of 5-dimensional hypo-contact solvable Lie groups. By solving hypo evolution equations and Hitchin evolution equations, I will show that all the resulting examples give rise to Ricci-flat metrics with holonomy $SU(3)$ and G_2. I will also present some recent results on 5-dimensional Lie groups admitting a left-invariant Sasakian structure.

In the second part of the talk I will consider a generalization of hypo-contact manifolds in dimension $2n+1$, which can be characterized in terms both of spinors and differential forms. These manifolds have a natural $SU(n)$-*structure* and in the real analytic case correspond to contact manifolds whose symplectic cone is Calabi-Yau. Then, I will consider circle actions that preserve the structure, and determine conditions for the contact reduction to carry an induced structure of the same type.

The Time Slice Axion in Perturbative Quantum Field Theory

Klaus Fredenhagen, Universität Hamburg

The initial value problem in quantum field theory can be expressed in terms of the time slice axiom (also called primitive causality). This axiom states that the observables which are localized in an arbitrarily small neighbourhood of a Cauchy surface generate all other observables. We prove that the axiom holds true in an enlarged algebra of free fields which contains in particular the Wick products of free fields, and that its validity is stable under deformation of a given theory. In particular it is valid in theories obtained from free theories by formal perturbation theory.

On holonomy of supermanifolds

Anton Galaev, Masarykova Univerzita

I will introduce holonomy groups and holonomy algebras for connections on locally free sheaves over supermanifolds. I get a one-to-one correspondence between parallel sections of these sheaves and holonomy-invariant vectors. As the special case, I consider the holonomy of linear connections on supermanifolds. I give examples of parallel geometric structures on supermanifolds and the corresponding holonomies. For Riemannian supermanifolds I prove an analog of the Wu theorem. Also I define Berger superalgebras and consider their examples.

Lorentzian, Conformal and Quasiconformal Geometries, Supersymmetry and Representation Theory

Murat Günaydin, Penn State

I will begin with a review of the geometries of $N = 2$ Maxwell-Einstein Supergravity Theories (MESGT) with symmetric target manifolds in $d = 5, 4$ and 3 spacetime dimensions and their underlying Euclidean Jordan algebras followed by a review of the proposal to define generalized causal space-times coordinatized by Jordan algebras whose automorphism, reduced structure and linear fractional groups can be identified with the rotation, Lorentz and conformal groups. The symmetry groups of $N = 2$ MESGTs defined by Euclidean Jordan algebras in five, four and three dimensions are simply the Lorentz, conformal and quasiconformal groups associated with them. Quasiconformal group actions were discovered rather recently and extend to all Lie groups. In particular, the quasiconformal realizations of different real forms of E_8 are their first known geometric realizations and leave invariant a "light-cone" with respect to a quartic distance function. I will also discuss the geometries of three infinite families of novel five dimensional unified MESGTs that are defined by Lorentzian Jordan algebras, whose scalar manifolds are , in general, not homogeneous. This will be followed by a discussion of the proposals that the four dimensional and three dimensional U-duality groups of $N = 2$ MESGTs with symmetric target spaces act as spectrum generating conformal and quasiconformal groups of the corresponding five and four dimensional MESGTs , respectively. The quantization of geometric quasiconformal realizations of noncompact groups lead directly to their minimal unitary representations, which are the analogs of singleton representations of symplectic groups. I will then explain a remarkable mapping between the Killing potentials that generate the isometries of $N = 2$ sigma models that couple to supergravity in harmonic superspace and the generators of minimal unitary representations of their isometry groups obtained from their quasiconformal realizations. This implies that the fundamental spectra of these theories form minimal unitary representations of their isometry groups. I will conclude with a discussion of open problems and future directions.

Extremals for the Sobolev inequality on the quaternionic Heisenberg group and the quaternionic contact Yamabe problem

Stefan Ivanov, Sofia University

We give partial solution of the quaternionic contact Yamabe problem on the quaternionic spheres and a complete one on the seven sphere. We show that the torsion of the Biquard connection is the only obstruction quaternionic contact structure to be locally isomorphic to a 3-Sasakian one. We describe explicitly non-negative extremals for the Sobolev inequality on the seven dimensional Hesenberg group and determine the best constant in the L^2 Folland-Stein embedding theorem. We define a curvature-type tensor invariant called quaternionic contact (qc) conformal curvature in terms of the curvature and torsion of the Biquard connection. The discovered tensor is similar to the Weyl conformal curvature in Riemannian geometry and to the Chern-Moser invariant in CR geometry. We show that a quaternionic contact manifold is locally qc conformal to the standard flat quaternionic contact structure on the quaternionic Heisenberg group, or equivalently, to the standard 3-sasakian structure on the sphere if and only if the qc conformal curvature vanishes.

Ricci flow unstable cell centered at the Kähler-Einstein metric on the twistor space of positive quaternion Kähler manifolds

Ryoichi Kobayashi, Nagoya University

The twistor space of positive quaternion Kähler manifolds naturally admits two 1-parameter families of Riemannian metrics, one is the family of canonical deformation metrics and the other is the family introduced by B. Chow and D. Yang in 1989. I will compare the behavior under the Ricci flow of these two families. It turns out that the family of (scaled) Chow-Yang metrics constitutes a Ricci flow unstable cell in the sense of Perelman's W-functional. Therefore, the family is foliated by the trajectories of the Ricci flow ancient solutions whose asymptotic soliton is the Kähler-Einstein metric. We pick up ancient solutions which realize the collapse of the twistor space where the base quaternion Kähler manifold shrinks faster. We then look at the asymptotic behavior of the covariant derivative of the curvature tensor along these solutions corresponding to the collapse (rescaled so that the base manifold converges and the fiber blows up). As an application, we propose a proof to the conjecture that any locally irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces.

Special Geometry, Black Holes and Instantons

Thomas Mohaupt, University of Liverpool

I will give an overview of the special geometry of vector multiplets for both Lorentzian and Euclidean space-time signature, and discuss applications thereof to black holes and instantons.

Skew-symmetric prolongations of Lie algebras and applications

Paul-Andi Nagy, University of Auckland

We compute the intersection $\Lambda^3 \cap (\Lambda^1 \otimes \mathfrak{g})$ where $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{so}(n)$. Applications include uniqueness/structure results for connections with skew-symmetric torsion including the flat case, Plücker-type embeddings and metric n-Lie algebras in Euclidean signature.
Building on related background we shall also look at the holonomy representation of metric connections with vectorial torsion.

$GL(2,\mathbb{R})$ geometry of ODEs

Paweł Nurowski, Uniwerzytet Warszawski

We define a class of ordinary differential equations whose solution spaces are naturally equipped with $GL(2,\mathbb{R})$ geometry. If the order of the differential equation is odd, $(2k+1)$, these geometries are special Weyl geometries with a conformal metric of signature $(k, k+1)$. The object that makes these Weyl geometries special is a conformal symmetric tensor of rank higer than 2. For example, in case of equations of order five, the corresponding $GL(2,\mathbb{R})$ geometry is associated with a conformal symmetric tensor of rank 3, which is related to the conformal metric by a certain algebraic identity. In the lecture we discuss the situation of order five in more detail, showing that the $GL(2,\mathbb{R})$ geometry in this case defines a unique $GL(2,\mathbb{R})$ connection with a skew symmetric torsion. This torsion is 'small' in the sense that it resides in the smallest nonzero irreducible representation of $GL(2,\mathbb{R})$. If time permits, paralels between this $GL(2,\mathbb{R})$ geometry and the recently introduced 'nearly integrable irreducible $SO(3)$ geometries in dimension five' will be indicated.

The Skew-torsion holonomy theorem and the full isometry group of naturally reductive spaces.

Carlos Olmos, Universidad Nacional de Córdoba

In the first part of the talk we will survey results related to normal and Riemannian holonomy. We will explain a new application of these geometric methods (joint work with

Silvio Reggiani): a Simons type holonomy theorem for 1-forms with values in a Lie algebra, which are totally skew-symmetric. We use this result to obtain the following theorem: Let M be naturally reductive compact homogeuos Riemannian space. Assume that M is locally irreducible and not a global rank one symmetric space. Then $Iso^0(M) = AffC^0(M)$ (affine transformations of M with respect to the canonical connection). Moreover, $Iso(M)$ is always contained in $AffC(M)$, unless M is isometric to a compact simple Lie group with the bi-invariant metric (the group $AffC(M)$ is standard to calculate). This explains conceptually why isotropy irreducible spaces can only be presented with the full isometry group (a question in Joseph Wolf classification)

Holonomy groups and the heterotic string

George Papadopoulos, King's College London

I shall review some aspects of the classification of supersymmeytric solutions of supergravity theories. The main focus will be on the N=1 d=4 supergravity and the supergravity associated with the heterotic string. In particular I shall introduce the connections that appear in these supergravity theories and I shall explain how the associated Killing spinor equations are solved in all cases

Properties of hyperkahler manifolds and their twistor spaces

Martin Roček, Stony Brook

We describe the relation between supersymmetric sigma-models on hyperkahler manifolds, projective superspace, and twistor space. We review the essential aspects and present a coherent picture with a number of new results (joint work with Ulf Lindstrom).

Symplectic connections and extrinsic symplectic symmetric spaces

Lorenz Schwachhöfer, Universität Dortmund

A torsion free connection on a symplectic manifold (M, ω) is called *symplectic* if ω is parallel or, equivalently, its holonomy group is contained in the linear symplectic group. We describe some aspects of the holonomy groups of such manifolds. We also discuss some results on *extrinsic symplectic symmetric spaces* which are symplectic submanifolds of a symplectic vector space such that the geodesic reflection at each point extends to a linear map on the ambient space.

Life with 2 SUSYs in $D = 1$; M-theory reduced on Calabi-Yau 5-folds

Kellogg S. Stelle, Imperial College

A framework for M-theory cosmology which preserves a minimal amount of supersymmetry is given by M-theory reduced to $D = 1$ on Calabi-Yau 5-folds. The reduction involves some intricate steps, but in the end a unified superspace formalism for the resulting D=1 non-linear sigma model emerges. The talk will present the construction and outlook for the inclusion of other features such as fluxes and quantum corrections.

Linear deformations of quaternionic-Kahler manifolds

Stefan Vandoren, Universiteit Utrecht

Using twistor space techniques and symplectomorphisms, we develop a framework to study linearized deformations of hyper-Kähler and quaternionic-Kähler metrics. We discuss several examples.

Deformations of associative submanifolds with boundary

Frederik Witt, Universität Regensburg

Let M be a compact smooth manifold of holonomy G_2. We prove that the space of infinitesimal associative deformations of a compact associative submanifold Y with boundary in a coassociative submanifold X is the solution space of an elliptic problem. Further, we compute its virtual dimension and discuss examples. The talk is based upon joint work with D. Gayet, math/0802.1283.