deformations of associative submanifolds with boundary

Frederik Witt

Universität Regensburg

holonomy groups and applications in string theory
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Deformations of associative submanifolds with boundary [math/08021283]

joint with
Damien Gayet (Lyon)

Hermitian spaces

$(\mathbb{C}^m, h)$ standard hermitian vector space

natural substructure: complex (hermitian) subspaces

real picture

$(\mathbb{C}^m, h) \leftrightarrow (\mathbb{R}^{2m}, g, J)$, J isometry with $J^2 = -\mathrm{Id}$

complex subspaces $\leftrightarrow$ real subspaces closed under J

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Kähler manifolds

(M^{2m}, g, J) looks $\sim (\mathbb{R}^{2m}, g, J)$ up to 2^{nd} order

natural substructure: complex submanifolds $Y \subset M$, i.e. TY closed under J

remark

complex manifolds **homologically volume minimising** [Federer]

Y^{2k} complex submanifold, $Y' \in [Y] \in H_{2k}(M)$

$$\Rightarrow \mathrm{vol}(Y') = \int_{Y'} \mathrm{vol}_{g_{|Y'}} \geq \mathrm{vol}(Y)$$

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octonians

$\mathbb{O} = \mathbb{R}\mathbf{1} \oplus \mathrm{Im}\,\mathbb{O}$ normed division algebra, not associative: in general

$$[x, y, z] = x \cdot (y \cdot z) - (x \cdot y) \cdot z \neq 0$$

real picture

$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O} + g+$ cross product $x \times y = \mathrm{Im}(\overline{y} \cdot x)$,

$$x \times y \perp x, y \quad x \times y = -y \times x \quad \|x \times y\|_g = \|x \wedge y\|_g$$

natural substructure

associative subspaces Y: closed under $\times$

$\Rightarrow \dim = 0, \mathbf{3}, 7$ and $[x, y, z] = 0$ for all $x, y, z \in Y$

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G_2–manifolds

$(M^7, g, \times)$ looks $\sim (\mathbb{R}^7, g, \times)$ up to 2^{nd} order

- **associative submanifolds**: $Y^3 \subset M^7$ with TY closed under $\times$
- **coassociative submanifolds**: $X^4 \subset M^7$ with $TX^\perp$ is associative

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homologically volume minimising?

deforming associatives with boundary conditions

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deformations of associative submanifolds

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calibrated submanifolds [Harvey–Lawson]

(M, g, τ) (compact) Riemannian manifold, $\tau \in \Omega^k(M)$.

- τ **calibration** iff for all $x \in M$, $U^k \subset T_xM$: $\tau_{|U} \leq \mathrm{vol}_{g_{|U}}$
- $Y \subset M$ **calibrated** iff $\tau_{|Y} = \mathrm{vol}_{g_{|Y}}$
- $d\tau = 0 \Rightarrow$ calibrated submanifolds are h.v.m.

examples

- Kähler case: $\omega(x, y) = g(Jx, y)$, $\frac{\omega^m}{m!} \leq \mathrm{vol}_{g_{|Y^{2m}}}$ [Wirtinger]
  calibrated submanifolds = complex submanifolds
- G_2 case: $\varphi(x,y,z) = g(x \times y, z)$
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associative submanifolds of $\mathbb{R}^7$

Split $\operatorname{Im}\mathbb{O} = \operatorname{Im}\mathbb{H} \oplus \mathbb{H} = \mathbb{R}^3 \oplus \mathbb{R}^4$, $\operatorname{Im}\mathbb{H} = \langle i, j, k \rangle$. When is

$$\operatorname{graph} f \subset \operatorname{Im}\mathbb{H} \oplus \mathbb{H}, \quad f : U \subset \operatorname{Im}\mathbb{H} \to \mathbb{H}$$

calibrated?

- $\mathcal{D}(f) = -i \cdot \frac{\partial f}{\partial x_1} - j \cdot \frac{\partial f}{\partial x_2} - k \cdot \frac{\partial f}{\partial x_3}$ **Dirac operator**
- $C : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ **triple cross product**

theorem [Harvey–Lawson]

$\operatorname{graph} f$ calibrated iff

$$\mathcal{D}(f) = C(f) = C\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$

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unbounded deformation problem

Y closed associative. Zariski tangent space of

$$\mathfrak{M}_Y = \{Y' \,|\, Y' \text{ associative and isotopic to } Y\}?$$

motivating example

$Y = \operatorname{Im}\mathbb{H} = \operatorname{graph} f \subset \mathbb{R}^7$ with $f \equiv 0$ calibrated, $\nu = \mathbb{H}$

$\Rightarrow$ f close to 0, linearised equation $\mathcal{D}(f) = 0$

theorem [McLean]

- normal bundle $\nu \to Y$ is a (twisted) spinor bundle for Y
- Zariski tangent space $= \ker \mathcal{D}$
  $\mathcal{D} : \Gamma(Y, \nu) \to \Gamma(Y, \nu)$ (twisted) Dirac operator
- virtual dimension of $\mathfrak{M}_Y = \operatorname{ind}(\mathcal{D}) = 0$

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- X coassociative
- Y compact associative with boundary $\partial Y \subset X$
- $\mathfrak{M}_{X,Y} = \{Y' \mid Y' \text{ associative isotopic to } Y, \partial Y' \subset X\}$

question

- what is the Zariski tangent space to $\mathfrak{M}_{X,Y}$?
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near the boundary

- $\nu \to Y$ normal bundle, $\nu_X \overset{\text{Def}}{=} T\partial Y^{\perp_{TX}}$
- $\mathcal{C} \subset Y$ collar neighbourhood of ∂Y, u inward pointing normal vector field
- $u\times : \nu_{|\mathcal{C}} \to \nu_{|\mathcal{C}}$ almost complex structure (cf. $u \in \mathrm{Im}\,\mathbb{H}$ acting on $\mathbb{H}$)

lemma [Gayet–W.]

- $\nu_X \subset \nu_{|\partial Y}$ and ν_X is $u\times$–closed
- $\mu_X \overset{\text{Def}}{=} \nu_X^{\perp_\nu}$ also $u\times$–closed
- $\overline{\mu}_X \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$ as $\mathbb{C}$–line bundles

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the Zariski tangent space

corollary

$\mathcal{D}\colon \Gamma(Y,\nu)\to\Gamma(Y,\nu)$ Dirac, $\mathcal{B}\colon \Gamma(\partial Y,\nu)\to\Gamma(\partial Y,\mu_X)$ proj
$\Rightarrow$ Zariski tangent space of $\mathfrak{M}_{X,Y}$ given by

$$\begin{array}{rcl} \mathcal{D}f & = & 0 \\ \mathcal{B}(f_{|\partial Y}) & = & 0 \end{array}$$

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- elliptic system?
- if yes, what is its index ($=$ virtual dimension of $\mathfrak{M}_{X,Y}$)?

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Calderón operator associated with $\mathcal{D}$

$$\mathcal{Q}_{\mathcal{D}} : \Gamma(\partial Y, \nu) \to \{ f_{|\partial Y} \in \Gamma(\partial Y, \nu) \,|\, \mathcal{D} f = 0 \}$$

definition

M^{2n+1} with boundary, $S \to M$ spinor bundle with Dirac $\mathcal{D} : \Gamma(M, S) \to \Gamma(M, S)$, $\mathcal{B} : \Gamma(\partial M, S) \to \Gamma(\partial M, V)$

$\mathcal{B}$ defines **local elliptic boundary condition** $\Leftrightarrow$ principal symbol $\sigma(\mathcal{B})$ satisfies

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- $\mathcal{B}$ defines l.e.b.c. then well–defined

$$\mathrm{ind}(\mathcal{D}, \mathcal{B}) = \dim\ker(\mathcal{D} \oplus \mathcal{B}) - \dim\mathrm{coker}(\mathcal{D} \oplus \mathcal{B})$$

- $\mathcal{P}^+ : \Gamma(M, S) \to \Gamma(\partial M, S^+)$ orthogonal projection on positive spinors

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∂Y connected $\Rightarrow \operatorname{ind}(\mathcal{D}, \mathcal{B}) = \int_{\partial Y} c_1(\nu_X) + 1 - g$
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sketch of the proof

- on $\partial Y \times [0,\epsilon)$ collar neighbourhood, we have

$$\mathcal{D} = u \times (\nabla_u + \mathcal{R})$$

- compute $\sigma(\mathcal{Q}_\mathcal{D})$ from $\sigma(\mathcal{R})$ (Calderón–Seeley)
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[compact examples]

use Joyce's construction of (co–)associatives to produce examples with non–vanishing index in **compact** holonomy G_2–manifolds

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