LORENTZIAN, CONFORMAL AND QUASICONFORMAL GEOMETRIES, SUPERSYMMETRY AND REPRESENTATION THEORY

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"WORKSHOP ON HOLONOMY GROUPS AND APPLICATIONS IN STRING THEORY"

LORENTZIAN GEOMETRY OF 5d, N=2 SUPERGRAVITY COUPLED TO N=2 VECTOR MULTIPLETS: MG, SIERRA + TOWNSEND (1983)

("very special real geometry")

$$\left(e_\mu^m + \psi_\mu^i + \underbrace{\overset{\circ}{A}_\mu\right) \oplus \left(A_\mu^a}_{A_\mu^I} + \lambda^{ai} + \phi^a\right)$$

$$e^{-1}\mathcal{L}_{bosonic} = -\tfrac{1}{2}R - \tfrac{1}{4}\overset{\circ}{a}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \tfrac{1}{2}g_{xy}\partial_\mu\varphi^x\partial^\mu\varphi^y$$
$$+\frac{e^{-1}}{6\sqrt{6}}C_{IJK}F^I_{\mu\nu}F^J_{\lambda\rho}A^K_\sigma\,\epsilon^{\mu\nu\lambda\rho\sigma}$$

$a, b = 1, \ldots n$

$I, J, \ldots = 0, 1, \ldots, n$

$x, y, \ldots = 1, \ldots, n$

CONSTANT SYMMETRIC TENSOR C_{IJK}

REMARKABLE FACT: N=2 Maxwell-Einstein supergravity (MESGT) IS UNIQUELY DETERMINED BY C_{IJK}

SCALAR MANIFOLD $\mathcal{M}$ IS AN HYPERSURFACE IN AN (n+1) DIMENSIONAL AMBIENT SPACE $\mathcal{C}$ WITH COORDINATES h^I AND METRIC a_{IJ}:

$$a_{IJ} = -\tfrac{1}{3}\partial_I\partial_J \ln \mathcal{N}(h)$$

$$\mathcal{N}(h) = C_{IJK}h^I h^J h^K$$

$\mathcal{M}$ is THE $\mathcal{N}(h) = 1$ HYPERSURFACE

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}(h)\Big|_{\mathcal{N}=1}$$

$$g_{xy} = \overset{\circ}{a}_{IJ}h^I_x h^J_y \qquad h^I_x \equiv \frac{\partial h^I}{\partial\varphi^x}$$

GLOBAL SYMMETRIES $\equiv$ SYMMETRIES OF C_{IJK}

ELEVEN DIMENSIONAL SUGRA COMPACTIFIED OVER A CY 3FOLD YIELDS A 5d N=2 MESGT COUPLED TO N=2 HYPERMULTIPLETS

C_{IJK} = TOPOLOGICAL INTERSECTION NUMBERS OF CY

$h_{1,1} = n_V + 1$ $\qquad$ $h_{2,1} = n_H - 1$

RIEMANN TENSOR OF SCALAR MANIFOLD $\mathcal{M}$ OF N = 2 MESGT

$$R_{xyzu} = \frac{4}{3}\left\{ g_{x[u}\, g_{z]y} + T_{x[u}{}^{w}\, T_{z]yw} \right\}$$

$$T_{xyz} = C_{IJK}\, h^I_x\, h^J_y\, h^K_z$$

IF T_{xyz} IS COVARIANTLY CONSTANT THEN $\mathcal{M}$ IS A SYMMETRIC SPACE

$$T_{xyz;u} = 0 \implies R_{xyzu;w} = 0$$

$T_{xyz;u} = 0$ $\oplus$ N=2 SUSY $\oplus$ POSITIVITY OF K.E IMPLY THAT $\mathcal{N}(h) = C_{IJK}\, h^I h^J h^K$ CAN BE IDENTIFIED WITH THE NORM FORM OF A EUCLIDEAN JORDAN ALGEBRA J OF DEGREE 3 !

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

$Str_0(J)$ = INVARIANCE GROUP OF THE NORM OF J
$\equiv$ REDUCED STRUCTURE GROUP OF J

$Aut(J)$ = AUTOMORPHISM GROUP OF J

JORDAN ALGEBRAS

$X \circ Y = Y \circ X \qquad X, Y \in J$

$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2$

EUCLIDEAN (FORMALLY REAL) JORDAN ALGEBRAS HAVE THE PROPERTY

$X^2 + Y^2 = 0 \implies X = 0$ AND $Y = 0$

$n \times n$ HERMITIAN MATRICES OVER $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$ FORM A JORDAN ALGEBRA J_n^A WITH THE PRODUCT

$$X \circ Y = \frac{1}{2}(XY + YX)$$

WHICH PRESERVES HERMITICITY.

COMPLETE LIST OF SIMPLE EUCLIDEAN JORDAN ALGEBRAS (FINITE DIMENSIONAL):

$J_n^{\mathbb{R}}, \quad J_n^{\mathbb{C}}, \quad J_n^{\mathbb{H}}$

$J_3^{\mathbb{O}}$ = 3x3 Hermitian over real octonions

$\Gamma(D) \equiv$ DIRAC GAMMA MATRICES IN D-DIM'NAL EUCLIDEAN SPACE

$J_3^{\mathbb{O}}$ HAS NO REALIZATION IN TERMS OF ASSOCIATIVE MATRICES WITH THE ABOVE PRODUCT !

$\iff$ EXCEPTIONAL JORDAN ALGEBRA

$\exists$ FOUR SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$J_3^{\mathbb{R}}, \quad J_3^{\mathbb{C}}, \quad J_3^{\mathbb{H}}, \quad J_3^{\mathbb{O}}$

AND AN INFINITE FAMILY OF NON-SIMPLE JORDAN ALGEBRAS OF DEGREE 3

$J = \mathbb{R} \oplus \Gamma(D)$

GENERALIZED SPACETIMES COORDINATIZED BY JORDAN ALGEBRAS

Nuovo Cimento 1975

MINKOWSKI SPACE: COORDINATES x_μ, METRIC $\eta_{\mu\nu}$

$x = x_\mu \sigma^\mu$ $\quad \sigma^\mu = (1, \sigma^i)$ $\quad i = 1,2,3$, Pauli Matrices

$x \in J_2^{\mathbb{C}}$ $\quad x \circ y \equiv \frac{1}{2}(xy + yx) \in J_2^{\mathbb{C}}$

$\mathrm{Aut}(J_2^{\mathbb{C}}) = SU(2)$

$N(x) = \mathrm{Det}\, x = \eta_{\mu\nu} x^\mu x^\nu, \quad \eta_{\mu\nu} = (+ - - -)$

$\mathrm{Str}_0(J_2^{\mathbb{C}}) = SL(2, \mathbb{C})$ = INVARIANCE GROUP OF NORM

$\text{Möb}(J_2^{\mathbb{C}}) = SU(2,2)$ = CONFORMAL GROUP

Möbius group ≡ linear fractional group (generated by (translations) + (inversions) + ($\mathrm{Str}(J) \equiv \mathrm{Str}_0(J) \times \mathcal{D}$))

FOR A GENERAL JORDAN ALGEBRA J WITH A BASIS e_I $(I = 1, \ldots, \dim(J))$ AND NORM FORM N

$x = e_I x^I$ $\quad \|x\| = N(x)$

$\mathrm{Aut}(J) \equiv$ Rotation Group (RG)

$\mathrm{Str}_0(J) \equiv$ Lorentz Grp (LG) (invariance group of N)

$\text{Möb}(J) \equiv$ Conformal Group (CG)

EXAMPLE: EXCEPTIONAL JORDAN ALGEBRA

	OVER REAL OCTONIONS $J_3^{\mathbb{O}}$	OVER SPLIT OCTONIONS $J_3^{\mathbb{O}_s}$
RG	F_4	$F_{4(4)}$
LG	$E_{6(-26)}$	$E_{6(6)}$
CG	$E_{7(-25)}$	$E_{7(7)}$

SYMMETRIES OF GEN. SPACETIMES DEFINED BY EUCLIDEAN (FORMALLY REAL) JORDAN ALGEBRAS J

J	RG(J)	LG(J)	Conf(J)
$\Gamma(d)$	$SO(d)$	$SO(d,1)$	$SO(d,2)$
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$

$J_n^{\mathbb{A}}$ = JA of $n\times n$ Hermitian matrices over the division algebra $\mathbb{A}$

$\Gamma(d)$ = Dirac gamma matrices in d dimensions

CONFORMAL GROUPS OF EUCLIDEAN JORDAN ALGEBRAS ALL ADMIT POSITIVE ENERGY UNITARY REPRESENTATIONS. $\Rightarrow$ CAUSAL SPACETIMES

CONFORMAL GROUP OF A JORDAN ALGEBRA J OF DEGREE P ACTS AS A "LINEAR FRACTIONAL GROUP" OF J. KOECHER

NORM FORM N INDUCES A SYMMETRIC P-FORM

$$\rightarrow \qquad N(X) \equiv N(\underbrace{X,\dots,X}_{P}) \quad \text{OVER } J.$$

Conf(J) LEAVES INVARIANT THE "MEASURE OF THE P ANGLE" FORMED BY P "STRAIGHT" LINES WITH DIRECTION VECTORS $\underbrace{X,Y,\dots,Z}_{P}$

$$\frac{N(X,Y,\dots,Z)^P}{N(X)\cdots\cdots N(Z)}$$

KANTOR

CONFORMAL GROUP OF J LEAVES INVARIANT THE CROSS RATIO

$$\frac{N(x-z)\,N(y-w)}{N(y-z)\,N(x-w)}$$ KANTOR

FOR $p=2$ THEY REDUCE TO THE STANDARD PROPERTIES OF CONFORMAL TRANSFORMATIONS.

THE LIE ALGEBRA $g(J)$ OF $Conf(J)$ HAS A 3-GRADED DECOMPOSITION

$$g(J) = g^{-1} \oplus g^{0} \oplus g^{+1}$$

g^{0} = structure algebra of J

$g^{-1} \Leftrightarrow J$ + $g^{+1} \Leftrightarrow \bar{J}$

$$g(J) = U_a \oplus S_{ab} \oplus \tilde{U}_b \qquad a,b \in J$$

Tits Kantor Koecher

$$[U_a, \tilde{U}_b] = S_{ab} \in g^{0}$$

$$[S_{ab}, U_c] = U_{\{abc\}} \in g^{-1}$$

$$[S_{ab}, \tilde{U}_c] = \tilde{U}_{\{bac\}} \in g^{+1}$$

$$[S_{ab}, S_{cd}] = S_{\{abc\}d} - S_{\{bad\}c}$$

$$\{abc\} \equiv a\circ(b\circ c) + c\circ(b\circ a) - (a\circ c)\circ b \qquad \text{JTP}$$

$$\{abc\} = \{cba\}$$

CONFORMAL REALIZATION: $x \in J$

$U_a(x) = a$ TRANSLATIONS

$S_{ab}(x) = \{abx\}$ LORENTZ GROUP × DILATATIONS

$\tilde{U}_c(x) = -\frac{1}{2}\{xcx\}$ SPECIAL CONFORMAL TRANSFORMATIONS

STRUCTURE GROUP = LORENTZ GROUP × DILATIONS

POINCARÉ GROUP OF J ≡ $STR(J)$ Ⓢ Translations

QUADRATIC MAP : $X \longrightarrow X^{\#}$

$$(X^{\#})_I = C_{IJK} X^J X^K$$

$$T_{xyz;u} = 0 \quad \Rightarrow \quad (X^{\#})^{\#} = N(X)\, X$$

ADJOINT IDENTITY

$$N(X) = C_{IJK} X^I X^J X^K$$

cubic norm of a degree 3 Jordan Algebra

SCALAR MANIFOLD of $N=2$ MESGT :

$$\mathcal{M} = \frac{Str_0(J)}{Aut(J)}$$

ADJOINT IDENTITY $\Longleftrightarrow$ LEGENDRE INVARIANT CUBIC POLYNOMIALS

Kazhdan & Etingof Polischuk (2000)

GENERIC JORDAN FAMILY : $J = \mathbb{R} \oplus \Gamma(Q)$

$$\mathcal{N} = \alpha Q$$

$\alpha \in \mathbb{R}$, Q is a quadratic form of signature $(+,-,-,\cdots)$

$$\mathcal{M} = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

FOR $N=2$ MESGT's DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 :

$$\mathcal{M}(J_3^{\mathbb{R}}) = \frac{SL(3,\mathbb{R})}{SO(3)} \qquad (5+1) \text{ vectors}$$

$$\mathcal{M}(J_3^{\mathbb{C}}) = \frac{SL(3,\mathbb{C})}{SU(3)} \qquad (8+1) \text{ vectors}$$

$$\mathcal{M}(J_3^{\mathbb{H}}) = \frac{SU^*(6)}{USp(6)} \qquad 14+1 \text{ vectors}$$

$$\mathcal{M}(J_3^{\mathbb{O}}) = \frac{E_{6(-26)}}{F_4} \qquad 26+1 \text{ vectors}$$

NORM FORM $\simeq$ 'DETERMINANT' OF J_3^A

N=2, 5d MESGT's DEFINED BY SIMPLE JORDAN ALGEBRAS OF DEGREE 3 ARE UNIFIED THEORIES

$$\xleftarrow{\text{SUSY}}\qquad\qquad\qquad\qquad \mathcal{M} = G/H$$

$$e_\mu^m \quad \psi_\mu^i \quad \begin{pmatrix} A_\mu^0 \\ A_\mu^a \end{pmatrix} \updownarrow G \quad \lambda^{ia} \quad \phi^a$$

$$\xleftarrow{\text{SUSY}}$$

UNIFIED MESGT $\Longleftrightarrow$ C_{IJK} = INVARIANT TENSOR OF A SIMPLE GROUP G

THEY ARE THE ONLY UNIFIED MESGT's IN $d=5$ WHOSE SCALAR MANIFOLDS ARE SYMMETRIC SPACES!

GST 1983

FURTHERMORE THERE EXIST 3 INFINITE FAMILIES OF UNIFIED MESGT'S IN $d=5$ WHOSE SCALAR MANIFOLDS ARE NOT HOMOGENEOUS SPACES (plus a sporadic one)

M.G + Zagermann 2003

C_{IJK} ≡ STRUCTURE CONSTANTS OF NONCOMPACT (LORENTZIAN) JORDAN ALGEBRAS OF DEGREE > 4

$$J^{\mathbb{R}}_{(1,N)_0}\ ,\quad J^{\mathbb{C}}_{(1,N)_0}\ ,\quad J^{\mathbb{H}}_{(1,N)_0}\ ,\quad N>3$$

$$\left(\text{plus}\quad J^{\mathbb{O}}_{(1,2)_0}\right)$$

GENERATED BY MATRICES OVER $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ THAT ARE HERMITIAN WITH RESPECT TO A LORENTZIAN METRIC

$$(X\eta)^\dagger = X\eta \qquad\qquad \eta = \mathrm{Diag}(+,-,-,\dots)$$

UNIFIED $N=2$ MESGT'S IN FIVE DIMENSIONS DEFINED BY LORENTZIAN JORDAN ALGEBRAS AND THEIR SYMMETRY GROUPS G $(N \neq 3)$

J	$G = Aut(J)$	# VECTOR FIELDS
$J^{\mathbb{R}}_{(1,N)_0}$	$SO(N,1)$	$\frac{1}{2}N(N+3)$
$J^{\mathbb{C}}_{(1,N)_0}$	$SU(N,1)$	$N(N+2)$
$J^{\mathbb{H}}_{(1,N)_0}$	$USp(2N,2)$	$N(2N+3)$
$J^{\mathbb{O}}_{(1,2)_0}$	$F_{4(-20)}$	26

$C_{IJK} \equiv d_{IJK}$ = structure constants of the traceless elements of $J^{\mathbb{A}}_{(1,N)}$

IN THE 5d, $N=2$, MESGT'S DEFINED BY EUCLIDEAN JORDAN ALGEBRAS OF DEGREE 3 C_{IJK} ARE GIVEN BY THE NORM FORM!

$N=2$ MESGT'S DEFINED BY $J^{\mathbb{C}}_3$, $J^{\mathbb{H}}_3$, $J^{\mathbb{O}}_3$ ARE EQUIVALENT TO THOSE DEFINED BY $J^{\mathbb{R}}_{(1,3)_0}$, $J^{\mathbb{C}}_{(1,3)_0}$ AND $J^{\mathbb{H}}_{(1,3)_0}$.

OF THE 3 INFINITE FAMILIES ONLY THE FAMILY DEFINED ALGEBRAS $J^{\mathbb{C}}_{(1,N)_0}$ CAN BE GAUGED SO AS TO OBTAIN AN INFINITE FAMILY OF UNIFIED YANG-MILLS-EINSTEIN SUPERGRAVITY THEORIES WITH GAUGE GROUPS $SU(N,1)$.

GEOMETRIES OF THE THEORIES DEFINED BY LORENTZIAN JORDAN ALGEBRAS?

GEOMETRIES OF 5d MESGT'S DEFINED BY JORDAN ALGEBRAS $J^{\mathbb{A}}_{(1,2)_0}$ $(\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O})$ AND CARTAN'S REMARKABLE ISOPARAMETRIC HYPERSURFACES :

$$J=\begin{pmatrix} -\xi^0+\xi^4 & \xi^3 & -\bar{\xi}^2 \\ \bar{\xi}^3 & -\xi^0-\xi^4 & -\xi^1 \\ \xi^2 & \bar{\xi}^1 & 2\xi^0 \end{pmatrix} \in J^{\mathbb{A}}_{(1,2)_0}, \quad \xi \in \mathbb{A},\ i=1,2,3, \quad \xi\ \xi \in \mathbb{R}$$

Scalar manifold of N=2 MESGT in d=5 is given by the hypersurface:

$$N(\xi) = d_{IJK}\,\xi^I\xi^J\xi^K = \frac{1}{3}\mathrm{Tr}\,J^3 = 1$$

$$\xi^I\Big|_{N=1} = h^I(\varphi^x)$$

$\mathrm{Aut}(J^{\mathbb{R}}_{(1,2)}) = SO(2,1)$, $\mathrm{Aut}(J^{\mathbb{C}}_{(1,2)}) = SU(2,1)$

$\mathrm{Aut}(J^{\mathbb{H}}_{(1,2)}) = USp(6,2)$, $\mathrm{Aut}(J^{\mathbb{O}}_{(1,2)}) = F_{4(-20)}$

THE SCALAR MANIFOLDS OF THESE THEORIES ARE FOLIATED BY THE NON-COMPACT ANALOGS OF CARTAN'S ISOPARAMETRIC HYPERSURFACES.

CARTAN'S HYPERSURFACES ARE DETERMINED BY THE CUBIC FUNCTION $\mathcal{F}$ DEFINED OVER EUCLIDEAN JORDAN ALGEBRAS: $\mathcal{F}=\frac{1}{3}\mathrm{Tr}(J^3)$, $J\in(J^{\mathbb{A}}_3)_0$

IN 4, 7, 13 AND 25 DIMENSIONAL SPHERES DEFINED BY THE CONDITION

$$\mathrm{Tr}\,J^2 = \text{constant}.$$

FOR LORENTZIAN JORDAN ALGEBRAS THE CORRESPONDING CONDITIONS ARE IN SPACES OF SPLIT SIGNATURES:

$(3,2)$, $(4,4)$, $(6,8)$, $(10,16)$

$\Downarrow$

AdS_4

SCALAR MANIFOLDS OF $N=2$ MESGT'S DEFINED BY $J^{\mathbb{R}}_{(1,3)_0}$, $J^{\mathbb{C}}_{(1,3)_0}$, $J^{\mathbb{H}}_{(1,3)_0}$ ARE SYMMETRIC SPACES

$$J^{\mathbb{R}}_{(1,3)_0} \Rightarrow \frac{SL(3,\mathbb{C})}{SU(2)}$$

$$J^{\mathbb{C}}_{(1,3)_0} \Rightarrow \frac{SU^*(6)}{USp(6)}$$

$$J^{\mathbb{O}}_{(1,3)_0} \Rightarrow \frac{E_{6(-26)}}{F_4}$$

FOR $N>3$, THE SCALAR MANIFOLDS OF THE THEORIES DEFINED BY $J^{\mathbb{A}}_{(1,N)_0}$ $(\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H})$ ARE "CUBIC SUBMANIFOLDS" OF COHOMOGENEITY 2 IN THE FOLLOWING SYMMETRIC SPACES:

$$\mathcal{M}_5(J^{\mathbb{R}}_{(1,N)_0}) \subset SL(N+1,\mathbb{R})/SO(N,1)$$

$$\mathcal{M}_5(J^{\mathbb{C}}_{(1,N)_0}) \subset SL(N+1,\mathbb{C})/SU(N,1)$$

$$\mathcal{M}_5(J^{\mathbb{H}}_{(1,N)_0}) \subset SU^*(2N+2)/USp(2N,2)$$

BY DIMENSIONAL REDUCTION TO $d=4$ AND TO $d=3$ SPACE-TIME DIMENSIONS ONE OBTAINS COMPLEX AND QUATERNIONIC MANIFOLDS. THEIR ISOMETRY GROUPS ARE IN GENERAL OF THE FORM:

$$\mathcal{M}_4 \quad : \quad Aut(J^{\mathbb{A}}_{(1,N)_0}) \times SO(1,1) ⓢ T_D$$

T_D = TRANSLATION GROUP

$$\mathcal{M}_3 \quad : \quad Aut(J^{\mathbb{A}}_{(1,N)_0}) \times SO(1,1)^2 ⓢ \mathcal{H}_{2D+3}$$

$\mathcal{H}$ = HEISENBERG GROUP

DIMENSIONAL REDUCTION OF 5d, N=2 MESGT

5d, N=2 MESGT WITH n VECTOR MULTIPLETS $\longleftrightarrow$ LORENTZIAN GEOMETRY
$\mathcal{M}_5$ = HYPERSURFACE IN A DOMAIN OF POSITIVITY DEFINED BY C_{IJK}.
(VERY SPECIAL REAL GEOMETRY)

$\Downarrow$

4d, N=2 MESGT WITH $(n+1)$ VECTOR MULTIPLETS $\longleftrightarrow$ CONFORMAL GEOMETRY
$\mathcal{M}_4$ = "UPPER HALF-PLANE" OF THE DOMAIN OF POSITIVITY
(VERY SPECIAL COMPLEX GEOMETRY)

$\Downarrow$

3d, N=4 SUGRA COUPLED TO $(n+2)$ N=4 MATTER MULTIPLETS $\longleftrightarrow$ QUASICONFORMAL GEOMETRY
(VERY SPECIAL QUATERNIONIC)

GENERIC JORDAN FAMILY OF N=2 MESGT'S

$$J = \mathbb{R} \oplus \Gamma(Q)$$

$$\mathcal{M}_5 = \frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$$

$$\mathcal{M}_4 = \frac{SO(n,2)}{SO(n) \times SO(2)} \times \frac{SU(1,1)}{U(1)}$$

$$\mathcal{M}_3 = \frac{SO(n+2,4)}{SO(n+2) \times SO(4)}$$

PURE N=2, d=5 SUGRA UNDER DIMENSIONAL REDUCTION YIELDS N=2, d=4 SUGRA COUPLED TO ONE VECTOR MULTIPLET WITH

$$\mathcal{M}_4 = \frac{SU(1,1)_G}{U(1)} \Rightarrow \left(F^I_{\mu\nu} \oplus \tilde{F}^I_{\mu\nu} \right) \sim s = \tfrac{3}{2} \text{ of } SU(1,1)_G$$

FURTHER REDUCTION TO d=3 YIELDS

$$\mathcal{M}_3 = G_{2(2)} / SU(2) \times SU(2)$$

SYMMETRY GROUPS OF N=2 MESGT's DEFINED BY SIMPLE (FR) JORDAN ALGEBRAS OF DEGREE 3 :

U-Duality	$J_3^{\mathbb{R}}$	$J_3^{\mathbb{C}}$	$J_3^{\mathbb{H}}$	$J_3^{\mathbb{O}}$
K_5	$SO(3)$	$SU(3)$	$USp(6)$	F_4
U_5	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
U_4	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
U_3	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

THE "MAGIC SQUARE" OF FREUDENTHAL, ROZENFELD AND TITS $\Rightarrow$ Magical Supergravity theories

	EXCEPTIONAL SUGRA $J_3^{\mathbb{O}}$	MAXIMAL SUGRA	COMMON SECTOR $= J_3^{\mathbb{H}}$ THEORY
$\mathcal{M}_5 =$	$E_{6(-26)}/F_4$	$E_{6(6)}/USp(8)$	$SU^*(6)/USp(6)$
$\mathcal{M}_4 =$	$E_{7(-25)}/E_6 \times U(1)$	$E_{7(7)}/SU(8)$	$SO^*(12)/U(6)$
$\mathcal{M}_3 =$	$E_{8(-24)}/E_7 \times SU(2)$	$E_{8(8)}/SO(16)$	$E_{7(-5)}/SO(12) \times SU(2)$

DISCRETE SUBGROUPS $E_{7(7)}(\mathbb{Z})$ and $E_{6(6)}(\mathbb{Z})$

$\Rightarrow$ SYMMETRY GROUPS OF NON-PERTURBATIVE SPECTRA OF M-THEORY TOROIDALLY COMPACTIFIED

T-DUALITY $\times$ S-DUALITY $\subset$ U-DUALITY

$SO(6,6) \times SL(2,\mathbb{R}) \subset E_{7(7)}$ Hull & Townsend

$SO(5,5) \times SO(1,1) \subset E_{6(6)}$

ANALOGOUS DECOMPOSITION FOR THE EXCEPTIONAL SUGRA

$SO(10,2) \times SL(2,\mathbb{R}) \subset E_{7(-25)}$

$SO(9,1) \times SO(1,1) \subset E_{6(-26)}$

GENERIC NON-JORDAN FAMILY OF $N=2$, $d=5$ MESGT'S WITH TARGET MANIFOLDS

$$\mathcal{M}_5 = SO(n,1)/SO(n)$$

MG, Sierra & Townsend 1986

IN THESE THEORIES ONLY THE PARABOLIC SUBGROUP

$$[SO(n-1) \times SO(1,1)] ⓢ T_{n-1} \subset SO(n,1)$$

IS A SYMMETRY OF THE LAGRANGIAN

(de Wit & van Proeyen, 1992)

CONSEQUENTLY, THE TARGET MANIFOLDS OF CORRESPONDING 4d THEORIES ARE HOMOGENEOUS NON-SYMMETRIC WITH ISOMETRY GROUPS:

$$G = [SO(2,1) \times SO(n-1) \times SO(1,1)] ⓢ \mathcal{H}^{2n-1}$$

$\mathcal{H}^D$ = HEISENBERG GROUP OF DIMENSION D.

UPON FURTHER REDUCTION TO 3 DIMENSION ONE OBTAINS $N=4$ SIGMA MODELS COUPLED TO SUPERGRAVITY WHOSE TARGET MANIFOLDS ARE HOMOGENEOUS NON-SYMMETRIC QUATERNIONIC.

THIS INFINITE FAMILY OF HOMOGENEOUS QUATERNIONIC MANIFOLDS WAS MISSING IN ALEKSEEVSKÏI'S CLASSIFICATION OF HOMOGENEOUS, NON-SYMMETRIC QUATERNIONIC MANIFOLDS.

de Wit & van Proeyen

V. Cortes

NOTE THAT ALEKSEEVSKÏI'S HOMOGENEOUS NON-SYMMETRIC SPACES ARE ALL VERY SPECIAL QUATERNIONIC i.e CAN BE OBTAINED FROM 5d, $N=2$ MESGT'S.

M/STRING THEORETIC ORIGINS OF N=2 MESGT'S DEFINED BY JORDAN ALGEBRAS, IN PARTICULAR THE MAGICAL THEORIES ?

DUAL PAIRS OF TYPE II STRING COMPACTIFICATION AS CONSTRUCTED BY SEN & VAFA (1995) USING METHODS DEVELOPED BY VAFA & WITTEN.

ORBIFOLDING $T^4 \times S^1 \times S^1 / \Gamma$:

→ $N=2$, $d=4$ THEORY WITH 15 MASSLESS VECTOR MULTIPLETS AND NO HYPERS CAN BE SHOWN TO BE THE MESGT DEFINED BY $J_3^{\mathbb{H}}$ WITH $\mathcal{M}_4 = \frac{SO^*(12)}{U(6)}$

THIS THEORY IS SELF-DUAL WITH THE DILATON BELONGING TO A VECTOR MULTIPLET.

→ GENERIC JORDAN THEORY WITH 7 VECTOR MUL MULTIPLETS

$$\mathcal{M}_4 = \frac{SO(6,2) \times SU(1,1)}{SO(6) \times U(1) \times U(1)} \qquad \text{NON-SELF-DUAL}$$

→ STU MODEL WITH 3 VECTOR AND 4 HYPER MULTIPLETS

$$\mathcal{M}_4 = \left[\frac{SU(1,1)}{U(1)}\right]^3 \times \frac{SO(4,4)}{SO(4) \times SO(4)} \qquad \text{SELF-DUAL}$$

→ $N=6$ SUPERGRAVITY HAS THE SAME SCALAR MANIFOLD AS THE $N=2$ MESGT DEFINED BY $J_3^{\mathbb{H}}$: $\mathcal{M}_4 = SO^*(12)/U(6)$ & IS SELF-DUAL

— M/SUPERSTRING THEORETIC ORIGIN OF THE EXCEPTIONAL SUPERGRAVITY DEFINED BY $J_3^{\mathbb{O}}$ WITH $\mathcal{M}_4 = \frac{E_{7(-25)}}{E_6 \times U(1)}$?

FHSV MODEL:

TYPE II ST ON A SELF-MIRROR CY WITH $h^{11} = h^{12} = 11$ DUAL TO HETEROTIC ST ON $K3 \times T^2$

QUANTUM MODULI $\equiv$ CLASSICAL MODULI

$$\mathcal{M}_4 = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_V = \frac{SO(10,2) \times SU(1,1)}{SO(10) \times U(1) \times U(1)} \qquad \mathcal{M}_H = \frac{SO(12,4)}{SO(12) \times SO(4)}$$

$\hookrightarrow$ MAXIMAL GENERIC JORDAN THEORY THAT SITS INSIDE THE EXCEPTIONAL MESGT.

$$SO(10,2) \times SU(1,1) \subset E_{7(-25)} \quad + \quad \not\subset E_{7(7)}$$

THERE EXISTS A UNIQUE ANOMALY-FREE SUGRA IN $d=6$ THAT REDUCES TO THE EXCEPTIONAL $N=2$ MESGT COUPLED TO 28 HYPERMULTIPLETS! MG + SEZGIN

$$\mathcal{M}_4 = \frac{E_{7(-25)}}{E_6 \times U(1)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)} = \mathcal{M}_V \times \mathcal{M}_H$$

$$\mathcal{M}_3 = \frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

MODULI SPACE OF FHSV MODEL IS THE SUBSPACE

$$\frac{SO(12,4)}{SO(12) \times SO(4)} \times \frac{SO(12,4)}{SO(12) \times SO(4)}$$

THE ABOVE OBSERVATIONS SUGGEST THAT THE EXCEPTIONAL MESGT COUPLED TO 28 HYPERS ON $E_{8(-24)} / E_7 \times SU(2)$ COULD BE OBTAINED FROM M-THEORY OVER AN EXCEPTIONAL CY WITH $h_{1,1} = h_{2,1} = 27$.

ORBITS OF BPS BLACK HOLES IN 5D, N=8 SUPERGRAVITY UNDER THE ACTION OF $E_{6(6)}$

FOR STATIONARY, SPHERICALY SYMMETRIC BPS BLACK HOLES WITH CHARGES q^I $(I=1,\ldots,27)$ ENTROPY IS GIVEN BY THE CUBIC INVARIANT

$$I_3 = C_{IJK}\, q^I q^J q^K \Rightarrow S = \sqrt{I_3}$$

$$I_3 \neq 0 \Longrightarrow \tfrac{1}{8} \text{ SUSY}$$

Ferrara + Maldacena (97)

$$I_3 = 0 + \frac{\partial I_3}{\partial q^I} \neq 0 \Rightarrow \tfrac{1}{4} \text{ SUSY}$$

$$I_3 = 0 = \frac{\partial I_3}{\partial q^I} \Rightarrow \tfrac{1}{2} \text{ SUSY}$$

ASSOCIATE WITH A BPS BH SOLUTION WITH CHARGES q^I AN ELEMENT OF SPLIT EXCEPTIONAL JORDAN ALGEBRA $J_3^{O_s}$ (FERRARA + MG 97)

$$J = \sum_{I=1}^{27} e_I\, J^I \qquad \{e_I\} \equiv \text{BASIS OF } J_3^{O_s}$$

$$\mathcal{N}(J) = I_3(q)$$

TIMELIKE ORBIT: $\mathcal{N}(J) > 0$

$$\frac{E_{6(6)}}{F_{4(4)}} = \frac{LG(J_3^{O_s})}{Aut(J_3^{O_s})} \qquad \tfrac{1}{8} \text{ BPS}$$

LIGHT-LIKE ORBIT:

$$\frac{E_{6(6)}}{O(5,4) \circledS T_{16}} \qquad \tfrac{1}{4} \text{ BPS}$$

CRITICAL LIGHT-LIKE ORBIT:

$$\frac{E_{6(6)}}{O(5,5) \circledS T_{16}} \qquad \tfrac{1}{2} \text{ BPS}$$

BPS AND NON-BPS ORBITS FOR EXTREMAL BH'S IN N=2 MESGT's IN d=5 AND d=4 WERE GIVEN IN 9708025 (Ferrara + MG)

SOLUTIONS TO ATTRACTOR EQUATIONS FOR NON-BPS ORBITS IN d=5 0606108 (FG)

$$V(\varphi^x, q_I) = q_I \overset{\circ}{a}{}^{IJ} q_J = Z^2 + \frac{3}{2} g^{xy} \partial_x Z \partial_y Z$$ BLACK HOLE POTENTIAL

$Z = q_I h^I$ = CENTRAL CHARGE

J	BPS ORBITS	NON-BPS ORBITS
$R + \Gamma$	$\frac{SO(n-1,1) \times SO(1,1)}{SO(n-1)}$	$\frac{SO(n-1,1) \times SO(1,1)}{SO(n-2,1)}$
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R}) / SO(3)$	$SL(3,\mathbb{R}) / SO(2,1)$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C}) / SU(3)$	$SL(3,\mathbb{C}) / SU(2,1)$
$J_3^{\mathbb{H}}$	$SU^*(6) / USp(6)$	$SU^*(6) / USp(4,2)$
$J_3^{\mathbb{O}}$	$E_{6(-26)} / F_4$	$E_{6(-26)} / F_{4(-20)}$

CRITICAL POINTS $\Rightarrow$ $\partial_x V \equiv \partial_{\varphi^x} V = 0$

$$\Rightarrow \quad 2Z\partial_x Z - \sqrt{\tfrac{3}{2}}\, T_{xyz}\, \partial^y Z \partial^z Z = 0$$

BPS

$$\partial_x Z = 0$$

$$V_{BPS}\Big|_{critical} = Z^2$$

$$\partial_x \partial_y V\Big|_{BPS} = \frac{8}{3} g_{xy} Z^2$$

Hessian > 0

NON-BPS

$$2Z\partial_x Z = \sqrt{\tfrac{3}{2}}\, T_{xyz}\, \partial^y Z \partial^z Z$$

$$V_{NON\text{-}BPS}\Big|_{critical} = 9Z^2$$

$$\partial_x \partial_y V_{NON\text{-}BPS} \geqslant 0$$

positive semi-definite

NOTE: N=8 SUGRA IN d=5 HAS ONLY ONE ORBIT WITH NON-ZERO ENTROPY

$$E_{6(6)} / F_{4(4)}$$

DEFINE THE DISTANCE BETWEEN TWO BH SOLUTIONS OF $N=8$ SUGRA IN $d=5$ (WITH CHARGES $q^I_A + q^I_B$) AS:

$$\mathcal{N}(J_A - J_B) \qquad J_A = q^I_A e_I \qquad J_B = q^I_B e_I$$

THE CUBIC DISTANCE FUNCTION IS INVARIANT UNDER THE GENERALIZED "POINCARÉ GROUP"

$$E_{6(6)} \circledS T_{27}$$

THE LIGHT-CONE WITH BASE POINT J_c DEFINED BY THE CONDITION

$$\mathcal{N}(J - J_c) = 0$$

IS INVARIANT UNDER THE CONFORMAL GROUP $E_{7(7)}$!

$E_{7(7)}$ ACTS AS A SPECTRUM GENERATING SYMMETRY IN $d=5$!!

FOR THE $N=2$ MESGT'S DEFINED BY EUCLIDEAN JORDAN ALGEBRAS OF DEGREE 3 THE ENTROPY OF BPS BH SOLUTIONS ARE ALSO GIVEN BY THE CUBIC NORM OF THE JA

$$s^2 = \mathcal{N}(J) \qquad J = e_I q^I$$

$\frac{1}{2}$ BPS BLACK HOLES OF THE EXCEPTIONAL SUGRA

$$J = e_I q^I \in \bar{J}_3^{\,0}$$

FOR $\mathcal{N}(J) > 0$ TWO POSSIBLE ORBITS

$$E_{6(-26)}/F_4 \qquad\qquad E_{6(-26)}/F_{4(-20)}$$

$$\lambda_1, \lambda_2, \lambda_3 > 0 \qquad\qquad \lambda_1 > 0 \; ; \; \lambda_2, \lambda_3 < 0$$

LIGHT-CONE WITH BASE POINT J_c DEFINED BY THE CONDITION $\mathcal{N}(J - J_c) = 0$ IS INVARIANT UNDER THE CONFORMAL GROUP $E_{7(-25)}$.

DIMENSIONAL REDUCTION TO $d=4$ MESGT's

$d=5$ $\quad A^I_\mu \Longleftrightarrow e^I \in J$.

$d=4$ $\quad F^I_{\mu\nu} \Longleftrightarrow e^I \quad , \quad \tilde{F}^{\mu\nu}_I \Longleftrightarrow e_I \in J$

A^0_μ from $d=5$ graviton $\quad F^0_{\mu\nu} \Leftrightarrow \mathbb{R} \quad , \quad \tilde{F}^{\mu\nu}_0 \Leftrightarrow \mathbb{R}$

$$\begin{pmatrix} F^0_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^{\mu\nu}_I & \tilde{F}^{\mu\nu}_0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \alpha & J \\ \tilde{J} & \beta \end{pmatrix} \in \mathcal{F}(J)$$

FREUDENTHAL TRIPLE SYSTEM DEFINED BY J

$\alpha, \beta \in \mathbb{R}$,

$X, Y, Z \in \mathcal{F}(J) \Rightarrow (X, Y, Z) \in \mathcal{F}(J)$

FREUDENTHAL TRIPLE PRODUCT

$\mathcal{F}(J)$ ADMITS A SYMPLECTIC INVARIANT FORM

$$\langle X, Y \rangle = - \langle Y, X \rangle \in \mathbb{R}$$

AND A QUARTIC INVARIANT

$$I(X) \equiv \langle (X, X, X), X \rangle$$

FOR THE $N=2$ MESGT's DEFINED BY JORDAN ALGEBRAS :

$\mathrm{Aut}\,\mathcal{F}(J) =$ U-DUALITY GROUP IN $d=4$

$U_4 = G_4 =$ ISOMETRY GROUP OF THE SCALAR MANIFOLD $\mathcal{M}_4$

$J = J_3^{\mathbb{O}}$ $\quad \mathrm{Aut}\,\mathcal{F}(J_3^{\mathbb{O}}) = E_{7(-25)}$

$$\mathcal{M}_4 = E_{7(-25)} / E_6 \times U(1)$$

ORBITS OF EXTREMAL BLACK HOLE SOLUTIONS IN $d=4$

$N=2$ MESGT + $N=8$ THEORY (MG + Ferrara 1997)

$(F^0_{\mu\nu}, F^I_{\mu\nu}) \oplus (\tilde{F}^{\mu\nu}_0, \tilde{F}^{\mu\nu}_I) \Leftrightarrow$ vector field strengths + their duals

$I = 1, \dots, 27$

$$\begin{pmatrix} F^0 & F^I \\ \tilde{F}_J & \tilde{F}_0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} p^0 & p^I \\ q_J & q_0 \end{pmatrix}$$

p = magnetic charges

q = electric charges

ASSOCIATE WITH A BLACK HOLE SOLUTION WITH ELECTRIC + MAGNETIC CHARGES AN ELEMENT OF $\mathcal{F}(J)$:

$$Q = \begin{pmatrix} p^0 & p^I e_I \\ q_I e^I & q_0 \end{pmatrix} \in \mathcal{F}(J)$$

THE ENTROPY OF THE BLACK HOLE IS GIVEN BY THE QUARTIC INVARIANT OF $\mathcal{F}(J)$

$$S^2 = I_4(Q) = I_4(p^0, p^I, q_0, q_I)$$

FOR $N=8$ SUPERGRAVITY, $J = J_3^{\mathbb{O}_s}$:

TIMELIKE ORBIT : $I_4 > 0$ $\quad E_{7(7)}/E_{6(2)}$

LIGHTLIKE ORBIT : $I_4 = 0$ $\quad E_{7(7)}/F_{4(4)} \circledS T_{26}$

CRITICAL LIGHT-LIKE ORBIT : $I_4 = 0$, $\partial I_4 = 0$ $\quad E_{7(7)}/O(6,5) \circledS \mathcal{H}^{33}$

DOUBLY CRITICAL LIGHT-LIKE ORBIT : $I_4 = 0$, $\partial I_4 = \partial\partial I_4\big|_{Adj} = 0$ $\quad E_{7(7)}/E_{6(6)} \circledS T_{27}$

$\mathcal{H}^{2n+1}$ = Heisenberg Algebra of dim. $2n+1$

$N=2$, $d=4$ MESGT's

EXTREMAL BLACK ATTRACTOR EQUATIONS

$\Rightarrow$ CRITICALITY CONDITIONS FOR THE BLACK HOLE POTENTIAL V_{BH}

$$V_{BH} \equiv |Z|^2 + G^{I\bar{J}} D_I Z \bar{D}_{\bar{J}} \bar{Z}$$

G = Kähler metric of $\mathcal{M}_4$

Z = covariantly holomorphic central charge function

$$Z = e^{\frac{1}{2}K(z,\bar{z})}\left[q_\Lambda X^\Lambda(z) - p^\Lambda F_\Lambda(z)\right]$$

For symmetric Kähler manifolds $\mathcal{M}_4$

$$D_i C_{jkl} = 0$$

Using the n_V-bein e_I^i one can switch to flat indices $I, J, \ldots, \bar{I}, \bar{J}, \ldots$

$$\partial_I V_{BH} = 0 \Longleftrightarrow 2\bar{Z} D_I Z + i\tilde{C}_{IJK} \delta^{J\bar{J}} \delta^{K\bar{K}} \bar{D}_{\bar{J}} \bar{Z} \bar{D}_{\bar{K}} \bar{Z} = 0$$

$\tilde{C}_{IJK}$ = $\widetilde{Str}(J)$ invariant tensor

$$X^\Lambda = \begin{pmatrix} X^0 \\ X^I \end{pmatrix} = \begin{pmatrix} 1 \\ z^I \end{pmatrix} \qquad z^I = A^I + i e^\sigma h^I$$

A^I = 4d scalar descending from 5d vectors

(A^0_μ, σ) = (vector + scalar) descending from 5d graviton

$$F_\Lambda = \partial_\Lambda F \qquad F = -\frac{\sqrt{2}}{3} \frac{C_{IJK} X^I X^J X^K}{X^0} = \text{prepotential}$$

$$g_{I\bar{J}} \equiv \partial_I \partial_{\bar{J}} K = \frac{3}{2} e^{-2\sigma} \overset{\circ}{a}_{IJ}$$

Kähler potential $K = -\ln\left[i \bar{X}^\Lambda F_\Lambda - i X^\Lambda \bar{F}_\Lambda\right]$

$$K = -\ln\left\{i \frac{\sqrt{2}}{3} C_{IJK} (z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K)\right\}$$

ORBITS OF EXTREMAL BH'S IN 4d, N=2 MESGT WITH $I_4 \neq 0$ (FG 97)

$\mathcal{F}(J)$	$\frac{1}{2}$BPS ($I_4 > 0$)	NON-BPS ($I_4 < 0$)	NON-BPS ($I_4 > 0$)
$\Gamma_{n,\mathbb{R}}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n+2)\times SO(2)}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n+1,1)\times SO(1,1)}$	$\frac{SO(n+2,2)\times SU(1,1)}{SO(n,2)\times SO(2)}$
J_3^{O}	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
J_3^{H}	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
J_3^{C}	$\frac{SU(3,3)}{SU(3)\times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\times SU(2,1)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$
	$Z \neq 0$, Hessian > 0	$Z \neq 0$, Hessian $\geqslant 0$	$Z = 0$, Hessian $\geqslant 0$

$$V_{BH} = |Z|^2 + G^{I\bar{J}} D_I Z \, \overline{D_J Z}$$

Bellucci, Ferrara, M.G + Marrani (2006)

Z = central charge

$$Z = e^{K(z,\bar{z})/2} \left(X^\Lambda(z) q_\Lambda - F_\Lambda(z) p^\Lambda \right) = L^\Lambda q_\Lambda - M_\Lambda p^\Lambda$$

$K(z,\bar{z})$ = Kähler potential , D_I = Kähler Covariant Derivative

CRITICAL POINTS :

$$\partial_I V_{BH} = 0 \Rightarrow 2\bar{Z} D_I Z + i C_{IJK} G^{J\bar{J}} G^{K\bar{K}} \overline{D_J Z}\, \overline{D_K Z} = 0$$

BPS $\Rightarrow$ $D_I Z = 0$ NON-BPS : $D_I Z \neq 0$

FOUR DIMENSIONAL U-DUALITY GROUP CONF(J) LEAVES THE LIGHT-CONE DEFINED BY THE CUBIC NORM $N(J)$ INVARIANT AND ACTS AS SPECTRUM GENERATING SYMMETRY GROUP OF FIVE DIMENSIONAL EXTREMAL BLACK HOLES!

$$\mathrm{Conf}(J) = K_J \oplus \underbrace{(LG \times \mathcal{D})}_{\text{LORENTZ GROUP + DILATIONS}} \oplus T_J$$

SPECIAL CONFORMAL TRANSFS. TRANSLATIONS

COULD THE 3-DIMENSIONAL U-DUALITY GROUP U_3 SIMILARLY ACT AS SPECTRUM GENERATING GROUP OF 4-DIMENSIONAL EXTREMAL BLACK HOLES?

MG, KOEPSELL + NICOLAI

PROBLEM: E_8, F_4 AND G_2 DO NOT ADMIT A 3-GRADING WITH RESPECT TO ANY SUBGROUP OF MAXIMAL RANK $\Rightarrow$ THEY CAN NOT BE REALIZED AS GENERALIZED CONFORMAL GROUPS!

EXCEPT FOR A_1, ALL SIMPLE LIE ALGEBRAS ADMIT A 5-GRADING W.R.T A SUBALGEBRA g^0 OF MAXIMAL RANK SUCH THAT $\dim g^{\mp 2} = 1$

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2}$$

$$g^{+1} \Updownarrow \mathcal{F}(J) \qquad g^{+2} \Updownarrow \mathbb{R}$$

SUCH LIE ALGEBRAS CAN BE REALIZED IN TERMS OF AN UNDERLYING FTS.

$$g^0 = \mathrm{Aut}(\mathcal{F}(J)) \oplus SO(1,1)$$

$$\mathrm{Aut}(\mathcal{F}(J)) \equiv U_4 = 4d \text{ U-DUALITY GROUP}$$

$$\approxeq \mathrm{Conf}(J)$$

ALL SIMPLE LIE ALGEBRAS INCLUDING G_2, F_4, E_8 ADMIT A 5-GRADED DECOMPOSITION w.r.t. A SUBALGEBRA g^0 OF MAXIMAL RANK

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2}, \qquad \dim g^{\pm 2} = 1$$

THEY CAN BE REALIZED OVER FREUDENTHAL-KANTOR TRIPLE SYSTEMS $\mathcal{F} \Longleftrightarrow$ SUBSPACE g^{+1} of g.

FREUDENTHAL'S CONSTRUCTION OF F_4, E_6, E_7, E_8

$$\mathcal{F}(J_3^{A}) = \begin{pmatrix} \alpha & J_3^{A} \\ \tilde{J}_3^{A} & \beta \end{pmatrix}$$

$A = \mathbb{R} \longleftrightarrow F_4$

$A = \mathbb{C} \longleftrightarrow E_6$

$A = \mathbb{H} \longleftrightarrow E_7$

$A = \mathbb{O} \longleftrightarrow E_8$

FREUDENTHAL TRIPLE PRODUCT:

$(X, Y, Z) \in \mathcal{F}(J)$

$\forall \quad X, Y, Z \in \mathcal{F}(J)$

Automorphism Groups of $\mathcal{F}(J_3^{A})$

$\mathrm{Aut}\, \mathcal{F}(J_3^{O_s}) \approx E_{7(7)}$

$\mathrm{Aut}\, \mathcal{F}(J_3^{O}) \approx E_{7(-25)}$

$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{H}}) \approx SO^*(12)$

$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{C}}) \approx SU(3,3)$

$\mathrm{Aut}\, \mathcal{F}(J_3^{\mathbb{R}}) \approx Sp(6, \mathbb{R})$

$\mathrm{Aut}\, \mathcal{F}(\mathbb{R} + \Gamma(d)) \approx SO(d,2) \times SO(2,1)$

$\mathcal{F}$ ADMITS A SYMPLECTIC INVARIANT FORM

$$\langle X, Y \rangle = - \langle Y, X \rangle \qquad X, Y \in \mathcal{F}$$

SUCH THAT THE QUARTIC INVARIANT I_4 OF $\mathcal{F}$ IS:

$$I_4(X) \equiv \langle (X, X, X), X \rangle$$

NOTE THAT $\mathrm{Aut}(\mathcal{F}(J_3)) \cong$ Conformal Group of J_3

SUBALGEBRA $g^0 = \mathrm{Aut}(\mathcal{F}(J)) \times SO(1,1)$

QUASICONFORMAL REALIZATION OF $E_{8(8)}$

M.G, KOEPSELL & NICOLAI (2000)

$$E_{8(8)} = \underbrace{\overset{x}{1} \oplus \overset{X}{56}} \oplus (E_{7(7)} + D) \oplus \tilde{56} \oplus \tilde{1}$$

$$E_{8(8)} = K \oplus U_A \oplus S_{AB} \oplus \tilde{U}_A \oplus \tilde{K}, \qquad A, B, \ldots \in \mathcal{F}(J)$$

CONSIDER THE ACTION OF $E_{8(8)}$ ON A 57 DIMENSIONAL SPACE WITH COORDINATES:

$$\mathbb{X} = (X, x) \qquad X \in \mathcal{F}(J_3^{\mathbb{O}_s}) \qquad x = \text{singlet}$$

$E_{8(8)}$ ACTION ON $\mathbb{X}$:

$$K(X) = 0 \qquad K(x) = 2a$$

$$U_A(X) = A \qquad U_A(x) = \langle A, X \rangle$$

$$S_{AB}(X) = (A, B, X) \qquad S_{AB}(x) = 2\langle A, B \rangle x$$

$$\tilde{U}_A(X) = \frac{1}{2}(X, A, X) - A x$$

$$\tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A \rangle + \langle X, A \rangle x$$

$$\tilde{K}(X) = -\frac{1}{6}(X, X, X) + X x$$

$$\tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X \rangle + 2x^2$$

(A, B, C) = FREUDENTHAL TRIPLE PRODUCT

$(U_A \oplus K)$ FORM A 57 DIMENSIONAL HEISENBERG ALGEBRA OF $E_{8(8)}$

$$[U_A, U_B] = \langle A, B \rangle K$$

SIMILARLY, $(\tilde{U}_A \oplus \tilde{K})$ FORM AN HEISENBERG ALGEBRA.

$\mathbb{X} = (\bar{X}, x)$, $\mathbb{Y} = (Y, y)$ $\quad \bar{X}, Y \in \mathcal{F}(J_3^{\mathbb{O}_s})$

DEFINE "SYMPLECTIC DIFFERENCE" $\ominus$ OF 57-VECTORS $\mathbb{X}$ + $\mathbb{Y}$ AS

$$\mathbb{X} \ominus \mathbb{Y} \equiv (\bar{X} - Y,\ x - y + \langle X, Y \rangle) = -\mathbb{Y} \ominus \mathbb{X}$$

AND THE QUARTIC NORM $\mathcal{N}_4$ OF A 57-VECTOR AS

$$\mathcal{N}_4(\mathbb{X}) \equiv 4\, I_4(X) - x^2$$

$\mathcal{N}_4$ IS MANIFESTLY INVARIANT UNDER $E_{7(7)}$.

LIGHT-CONE WITH BASE POINT $\mathbb{Y}_B$ DEFINED BY THE SET OF 57-VECTORS $\mathbb{X}$

$$\mathcal{N}_4(\mathbb{X} \ominus \mathbb{Y}_B) = 0$$

IS INVARIANT UNDER THE ABOVE ACTION OF $E_{8(8)}$.

$\Rightarrow$ FIRST KNOWN GEOMETRIC REALIZATION OF $E_{8(8)}$ AS THE INVARIANCE GROUP OF A LIGHT-CONE IN 57 DIMENSIONS DEFINED BY A QUARTIC DISTANCE FUNCTION.

$\Rightarrow$ THIS REALIZATION EXTENDS TO COMPLEX E_8 AND HENCE TO ALL ITS REAL FORMS!

QUASICONFORMAL REALIZATIONS OF OTHER SPLIT EXCEPTIONAL GROUPS

$$E_{7(7)} = \bar{1} \oplus \overline{32} \oplus (SO(6,6) + D) \oplus \underbrace{32 \oplus 1}$$

$$E_{6(6)} = \bar{1} \oplus \overline{20} \oplus (SL(6,\mathbb{R}) + D) \oplus \underbrace{20 \oplus 1}$$

$$F_{4(4)} = \bar{1} \oplus \overline{14} \oplus (Sp(6,\mathbb{R}) + D) \oplus \underbrace{14 \oplus 1}$$

$$G_{2(2)} = \bar{1} \oplus \bar{4} \oplus (SL(2,\mathbb{R}) + D) \oplus \underbrace{4 \oplus 1}$$

IDENTIFY THE CHARGE-ENTROPY SPACE OF EXTREMAL BLACK HOLES OF N=8 SUGRA WITH THE 57 DIMENSIONAL SPACE ON WHICH $E_{8(8)}$ ACTS AS THE QUASICONFORMAL GROUP

$$\mathbb{Q} = (Q_A, s) \qquad Q_A = (q_0, q_I, p^0, p^I) \qquad I = 1, \cdots, 27$$

s = entropy $\qquad Q_A$ = charges

$$\mathcal{N}_4(\mathbb{Q}) = I_4(q,p) - s^2$$

LIGHT-CONE: $\mathcal{N}_4(\mathbb{Q}) = 0 \Rightarrow s^2 = I_4(p,q)$

DEFINE THE DISTANCE BETWEEN TWO BLACK HOLE SOLUTIONS IN CHARGE-ENTROPY SPACE AS

$$d(\mathbb{Q}, \tilde{\mathbb{Q}}) = \mathcal{N}_4(\mathbb{Q} \ominus \tilde{\mathbb{Q}})$$

LIGHT-LIKE SEPARATIONS ARE LEFT INVARIANT UNDER THE QUASICONFORMAL ACTION OF $E_{8(8)}$.

PROPOSAL: $E_{8(8)}$ ACTS AS SPECTRUM GENERATING SYMMETRY GROUP OF EXTREMAL BLACK HOLES OF N = 8 SUGRA IN $d=4$ GKN

NOTE THAT $E_{8(8)} \approx U_3$ = U-DUALITY GROUP OF N=8 SUGRA IN $d=3$

RECALL THE PROPOSAL THAT $U_4 = E_{7(7)}$ ACTS AS THE SPECTRUM GENERATING CONFORMAL GROUP IN 5D, N=8 SUGRA.

SIMILARLY, THE U-DUALITY GROUPS U_3 OF N=2 MESGT'S DEFINED BY JORDAN ALGEBRAS J OF DEGREE 3 SHOULD ACT AS SPECTRUM GENERATING QUASICONFORMAL GROUPS OF THE CORRESPONDING N=2 MESGT'S IN $d=4$.

THE MINIMAL UNITARY REPRESENTATION OF $E_{8(8)}$

OBTAINED BY QUANTIZATION OF THE QUASICONFORMAL REALIZATION : GKN 2000

$$E_{8(8)} = E \oplus \begin{pmatrix} E_{ij} \\ E^{ij} \end{pmatrix} \oplus (E_{7(7)} + D) \oplus \begin{pmatrix} F_{ij} \\ F^{ij} \end{pmatrix} \oplus F$$

$$248 = 1 \oplus (28+\widetilde{28}) \oplus (133+1) \oplus (28+\widetilde{28}) + 1$$

28 COORDINATES $X^{ij} = -X^{ji}$ $(i,j = 1,..8)$

28 MOMENTA $P_{ij} = -P_{ji}$

$$[X^{ij}, P_{kl}] = i\, \delta^{ij}_{kl}$$ SL(8,ℝ) Basis

$$\begin{pmatrix} E^{ij} \\ E_{ij} \end{pmatrix} = \begin{pmatrix} x\, X^{ij} \\ x\, P_{ij} \end{pmatrix} \in g^{-1} \quad \Rightarrow \quad E = \frac{1}{2}x^2 \in g^{-2}$$

$$(G^i_j \oplus G^{ijkl}) \in E_{7(7)} \in g^0$$

$$G^i_j = 2X^{ik}P_{kj} + \frac{1}{4} X^{kl}P_{kl}\,\delta^i_j \qquad SU(8)$$

$$G^{ijkl} = -\frac{1}{2} X^{[ij} X^{kl]} + \frac{1}{48} \epsilon^{ijklmnpq} P_{mn} P_{pq} \qquad \frac{E_{7(7)}}{SU(8)}$$

MOMENTUM p CONJUGATE TO x $\quad [x, p] = i$

$$\begin{pmatrix} F^{ij} \\ F_{ij} \end{pmatrix} \in g^{+1} \qquad F \in g^{+2}$$

$$F^{ij} = -p\,X^{ij} + \frac{2i}{x}\left[X^{ij}, I_4(X,P)\right]$$

$$F_{ij} = -p\,P_{ij} + \frac{2i}{x}\left[P_{ij}, I_4(X,P)\right]$$

$$F = \frac{p^2}{2} + \frac{2 I_4}{x^2}, \qquad D = \frac{1}{2}(xp + px)$$

UNITARY REPRESENTATION ON THE HILBERT SPACE OF SQUARE INTEGRABLE FUNCTIONS IN 29 VARIABLES $(X_{ij}, x) \Rightarrow$ MINIMAL UIR OF $E_{8(8)}$.

$SL(2,\mathbb{R})$ IN THE MINIMAL UIR OF $E_{8(8)}$

$$E = \frac{1}{2}x^2 \quad , \quad D = \frac{1}{2}(xp+px) \quad , \quad F = \frac{p^2}{2} + \frac{2I_4}{x^2}$$

CONFORMAL QUANTUM MECHANICS

$I_4(X,P) \Longleftrightarrow$ COUPLING CONSTANT g

$SL(2,\mathbb{R})$ AND $E_{7(7)}$ FORM A DUAL PAIR IN $E_{8(8)}$

QUADRATIC CASIMIRS :

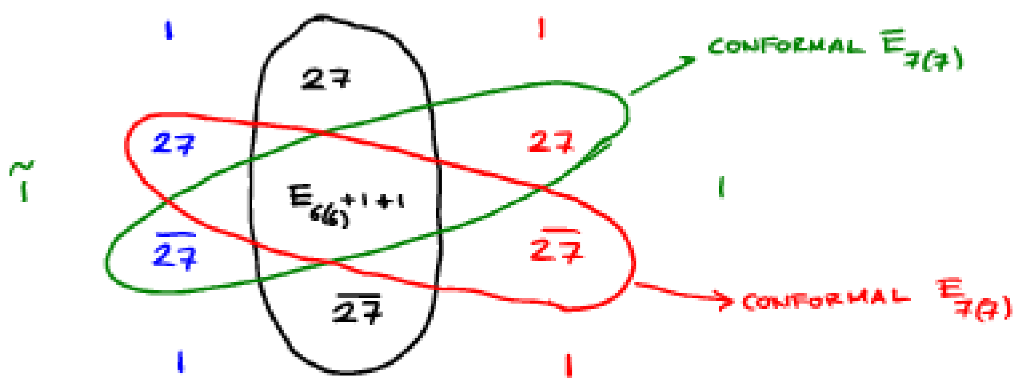
$$\mathcal{C}_2(SL(2,\mathbb{R})) = I_4 - \frac{3}{16}$$

$$\mathcal{C}_2(E_{7(7)}) = 3I_4 - \frac{969}{16}$$

$$\mathcal{C}_2(E_{8(8)}) = -120$$

MINIMAL UIR OF $E_{8(8)}$ DECOMPOSES INTO INFINITELY MANY IRREPS OF $E_{7(7)} \times SL(2,\mathbb{R})$.

$$E_{8(8)} = \tilde{1} \oplus \widetilde{56} \oplus (E_{7(7)} + D) \oplus 56 \oplus 1$$



$E_{7(7)}$ ACTS LINEARLY ON $g^{+1} = 56$

$E_{7(7)}$ CONFORMAL OVER 27 COORDINATES

$E_{7(7)}$ CONFORMAL OVER $\widetilde{27}$ MOMENTA

NOTE CONFORMAL REALIZATION VERSUS MINIMAL UIR :

$$E_{7(7)} = \tilde{1} \oplus \widetilde{32} \oplus (SO(6,6)+D) \oplus 32 \oplus 1$$

$32 = 16 + \widetilde{16} \Rightarrow$ MINREP OVER $L^2(16+1)$.

QUASICONFORMAL REALIZATION OF THE U-DUALITY GROUPS U_3 OF $N=4$ MESGT'S IN $d=3$ DEFINED BY JORDAN ALGEBRAS OF DEGREE 3

M.G. + PAVLYK

SCALAR MANIFOLDS ARE QUATERNIONIC SYMMETRIC

$$\mathcal{M}_3(\mathbb{R}+\Gamma(n)) = \frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$$

$$M_3(J_3^{\mathbb{R}}) = \frac{F_{4(4)}}{Usp(6)\times SU(2)} \qquad \mathcal{M}_3(J_3^{\mathbb{C}}) = \frac{E_{6(2)}}{SU(6)\times SU(2)}$$

$$M_3(J_3^{\mathbb{H}}) = \frac{E_{7(-5)}}{SO(12)\times SU(2)} \qquad M_3(J_3^{\mathbb{O}}) = \frac{E_{8(-24)}}{E_7\times SU(2)}$$

NOTE $$M_3(J) = \frac{QConf(J)}{\widetilde{Conf}(J)\times SU(2)}$$

$\widetilde{Conf}(J)$ = COMPACT REAL FORM OF $Conf(J)$

$QConf(J) \Rightarrow$ SPECTRUM GENERATING SYMMETRY OF 4D, N=2 MESGT

MINIMAL UIR'S OF $QConf(J)$ OBTAINED BY QUANTIZING THE GEOMETRIC ACTION (M.G + Pavlyk)

$$E_{8(-24)} = 1 \oplus 56 \oplus \left(E_{7(-25)} + D\right) \oplus 56 \oplus 1$$

$$E_{8(8)} = 1 \oplus 56 \oplus \left(E_{7(7)} + D\right) \oplus 56 \oplus 1$$

$$E_{8(8)} \supset SO(16) \qquad E_{8(-24)} \supset E_7\times SU(2)$$

$$E_{7(7)} \supset SU(8) \qquad E_{7(-25)} \supset SU(6,2)$$

$$E_{7(7)} \supset SL(8,\mathbb{R}) \qquad E_{7(-25)} \supset SU^*(8)$$

THE METHOD OF OBTAINING THE MINIMAL UNITARY REPRESENTATION BY QUANTIZATION OF QUASICONFORMAL REALIZATION EXTENDS TO ALL NON-COMPACT GROUPS AND SUPERGROUPS (M.G + Pavlyk)

$$QConf = K \oplus U_\Lambda \oplus (S_{(\Lambda\Sigma)} + \Delta) \oplus \tilde{U}_\Lambda \oplus \tilde{K}$$

$$K = \frac{y^2}{2} \qquad \Delta = \frac{1}{2}(yp+py) \qquad \tilde{K} = \frac{p^2}{2} + \frac{2I_4}{y^2}$$

$\{K, \Delta, \tilde{K}\}$ GENERATE $SL(2,\mathbb{R})$ OF CONFORMAL QUANTUM MECHANICS (Fubini et.al)

$[y, p] = i \qquad I_4 = I_4(X, P)$ = coupling constant

$$U_\Lambda = \begin{pmatrix} y X_A \\ y P^A \end{pmatrix} \qquad [X_A, P^B] = i\delta_A^B$$

$I_4 \Rightarrow C_2(H)$ = quadratic Casimir of H generated by $S_{(AB)}$.

$S_{[\Lambda\Sigma]} \propto \Omega_{\Lambda\Sigma}\,\Delta \qquad \Omega_{\Lambda\Sigma}$ = Symplectic metric of H.

FOR $QConf(J)$: $\quad S_{(\Lambda\Sigma)} \overset{\sim}{=} Conf(J)$

$$Conf(J) = R_I \oplus (R_I^J + \mathcal{R}) \oplus \tilde{R}^I = Aut\mathcal{F}(J)$$

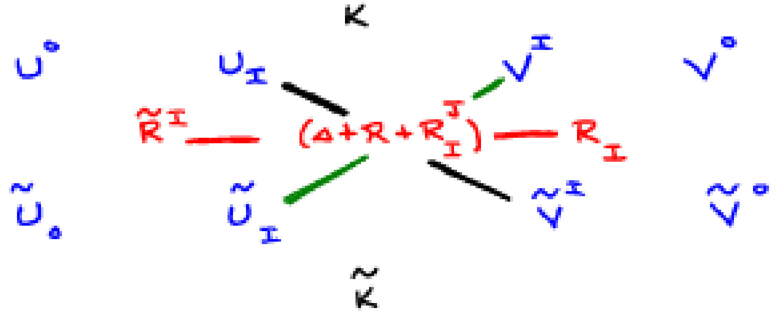
$QConf(J)$ has 7-grading w.r.t $\mathcal{R}$

$$X_A = (q_0, q_I) \qquad P^A = (p^0, p^I) \quad , \quad J = e^I q_I$$

$$I_4(X, P) = p^0 N(q) + q_0 N(p) + (p^0 q_0 - p^I q_I)^2 + q^I_\# \, p^\#_I$$

I_4 differs from the quadratic Casimir of $Conf(J)$ by a c-number due to normal ordering!

WITH RESPECT TO $(\mathcal{R}, \Delta)$ THE LIE ALGEBRA OF $QCon(J)$ HAS A $(7,5)$-GRADED STRUCTURE



$R_I^I = 0 \rightarrow R_I^J$ = GENERATORS OF LORENTZ GROUP OF $J \cong U_5$

MINIMAL UNITARY REALIZATION OF $QCONF(J)$ OVER THE HILBERT SPACE OF SQUARE INTEGRABLE FUNCTIONS IN $D = \dim(J) + 2$ VARIABLES:

$Conf(J)$ GENERATED BY $R_I, \tilde{R}^J, [R_I, \tilde{R}^J]$ IS REALIZED AS BILINEARS OF POSITION (q_I, q_0, y) AND MOMENTUM (p^I, p^0, p) OPERATORS.

$CONF(J)$ GENERATED BY $U_I, \tilde{V}^J$ AND $[U_I, \tilde{V}^J]$ IS REALIZED NON-LINEARLY.

SIMILARLY, $CONF(J)$ GENERATED BY $\tilde{U}_I, V^J$ AND $[\tilde{U}_I, V^J]$ IS ALSO REALIZED NONLINEARLY.

$QConf(J_3^{\mathbb{O}}) = E_{8(-24)}$	$QConf(J_3^{\mathbb{O}_s}) = E_{8(8)}$
$Conf(J_3^{\mathbb{O}}) = E_{7(-25)}$	$Conf(J_3^{\mathbb{O}_s}) = E_{7(7)}$
$Lor(J_3^{\mathbb{O}}) = E_{6(-26)}$	$Lor(J_3^{\mathbb{O}_s}) = E_{6(6)}$
$Aut(J_3^{\mathbb{O}}) = F_4$	$Aut(J_3^{\mathbb{O}_s}) = F_{4(4)}$

QUANTUM ATTRACTOR FLOWS AND $U_3 = G_3$ AS SPECTRUM GENERATING SYMMETRY GROUP OF SSS BPS BLACK HOLES OF 4d MESGT's

M.G., Neitzke, Pioline + Waldron

0512296, 0707.0267

Attractor equations for a SSS BPS BH in $d=4, N=2$ MESGT $\approx$ Geodesic motion of a fiducial particle on the scalar manifold $\mathcal{M}_3^*$ of the 3d sugra obtained by reduction on a time-like circle.

Breitenlohner
Gibbons
Maison (1988)

d = 4 $\quad \mathcal{M}_4 = \dfrac{Conf(J)}{\widetilde{Str}(J)}$

spacelike circle $\swarrow$ $\qquad\qquad$ $\searrow$ timelike circle

$$\mathcal{M}_3 = \frac{QConf(J)}{\widetilde{Conf}(J) \times SU(2)} \qquad\qquad \mathcal{M}_3^* = \frac{QConf(J)}{Conf(J) \times SU(1,1)}$$

quaternionic Kähler $\qquad\qquad$ para-quaternionic Kähler

(RADIAL) QUANTIZATION OF THE GEODESIC MOTION ON $\mathcal{M}_3^*$

$\Rightarrow$ HILBERT SPACE $\equiv L^2(\mathcal{M}_3^*)$

GENERAL PHASE SPACE $\equiv$ TANGENT SPACE $T(\mathcal{M}_3^*)$

$\Rightarrow$ UNITARY REALIZATION OF $QConf(J)$ OVER $L^2(\mathcal{M}_3^*)$

BPS PHASE SPACE $\Leftrightarrow$ TWISTOR SPACE $\mathcal{Z}_3^*$ OF $\mathcal{M}_3^*$

BPS HILBERT SPACE IS GIVEN BY QUANTIZATION OF THE TWISTOR SPACE $\mathcal{Z}_3^*$

QUANTUM HILBERT SPACE OF BPS SSS BLACK HOLES $\Rightarrow$ UNITARY REPRESENTATIONS OF $\widetilde{QConf}(J) = G_3 = U_3$

N=2 SUPERSYMMETRIC SIGMA MODELS THAT COUPLE TO N=2 SUPERGRAVITY IN HARMONIC SUPERSPACE

TARGET SPACE IS A QUATERNIONIC KÄHLER MANIFOLD (BAGGER + WITTEN 1983)

HSS FORMULATION (GALPERIN + OGIEVETSKY 1992)

$$S = \int d\zeta^{-4} du \left\{ Q^{+}_{\alpha} D^{++} Q^{+\alpha} - q^{+}_{i} D^{++} q^{+i} + \mathcal{L}(Q^{+}, q^{+}, u^{-}) \right\}$$

$\zeta^{M} = \{ x^{A}_{\mu}, \theta^{\alpha +}, \bar{\theta}^{\dot{\alpha}+} \}, u^{\pm}_{i}$ ARE COORDINATES OF ANALYTIC SUPERSPACE: $u^{+i} u^{-}_{i} = 1 \quad i=1,2$

$$D^{++} u^{-}_{i} = u^{+}_{i}$$

$Q^{+}_{\alpha}(\zeta, u)$, $\alpha = 1, \ldots, 2n$ hypermultiplets

$q^{+}_{i}(\zeta, u)$, $i = 1,2$ sugra hypermultiplet compensators

THE ACTION S INVOLVES A SINGLE DERIVATIVE D^{++} AND HAS THE FORM OF HAMILTONIAN MECHANICS

$$\mathcal{L}^{+4} = \frac{P^{+4}(Q)}{(q^{+}u^{-})^{2}} \quad , \quad P^{+4} = \frac{1}{12} S_{\alpha\beta\gamma\delta} Q^{+\alpha} Q^{+\beta} Q^{+\gamma} Q^{+\delta}$$

$S_{\alpha\beta\gamma\delta}$ IS SYMMETRIC

ISOMETRIES OF THE TARGET MANIFOLD ARE GENERATED BY KILLING POTENTIALS K^{++}_{A} WHICH OBEY THE THE CONSERVATION LAW:

$$\partial^{++} K^{++}_{A} + \{ K^{++}_{A}, \mathcal{L}^{+4} \} = 0 \qquad \partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}$$

+ GENERATE THE LIE ALGEBRA OF THE ISOMETRY GROUP

$$\{ K^{++}_{A}, K^{++}_{B} \} = f_{AB}{}^{C} K^{++}_{C} \qquad \{ , \} = \text{P.B.}$$

CONSIDER SYMMETRIC TARGET SPACES

$G/H\times SU(2)$ $\qquad K^{++}_{A} \Leftrightarrow$ GENERATORS OF G

REMARKABLE MAPPING BETWEEN HSS FORMULATION OF N=2 SIGMA MODEL THAT COUPLES TO SUGRA AND THE MINIMAL UNITARY REALIZATIONS OF THEIR ISOMETRY GROUPS M.G. 2007

$$\mathcal{M} = G/H \times SU(2)$$

HSS FORMULATION	MINIMAL UNITARY REALIZATION OF G
$x_c = (q^{+i} u_i^-),\ (q^{+i} u_i^+) = p_c$	x, p
$Q^{+\alpha}\ (\alpha = 1, \cdots, 2n)$	$\xi^\alpha = (X_A, P^B)\quad A,B = 1,\cdots,n$
$\{\ ,\ \}_{P.B}$	$i[\ ,\ ]$
$\mathcal{P}^{+4}(Q)$	$I_4(X,P)$
$J_a^{++} = (t_a)_{\alpha\beta} Q^{+\alpha} Q^{+\beta}$	$J_a = (t_a)_{\alpha\beta} \xi^\alpha \xi^\beta$
$K = x_c^2$	$K = x^2$
$\tilde{K} = p_c^2 - \frac{2\mathcal{P}^{+4}}{x_c^2}$	$\tilde{K} = p^2 - \frac{I_4(X,P)}{y^2}$
$K_\alpha = x_c Q_\alpha^+$	$K_\alpha = x \xi_\alpha$
$\tilde{K}_\alpha = [\tilde{K}, K_\alpha]$	$\tilde{K}_\alpha = [\tilde{K}, K_\alpha]$
$\Delta = y_c p_c + p_c y_c$	$\Delta = yp + py$

$$\mathfrak{g} = K \oplus K_\alpha \oplus (J_a + \Delta) \oplus \tilde{K}_\alpha \oplus \tilde{K}$$

FUNDAMENTAL SPECTRUM OF THE QUANTUM N=2 SIGMA MODEL MUST BELONG TO THE MINIMAL UIR OF ITS ISOMETRY GROUP G.

FULL QUANTUM SPECTRUM MUST BELONG TO THE REPRESENTATIONS FORMED BY TENSORING OF MINIMAL UIR OF G!

FOR $N=2$ MESGT'S $QConf(J)$ IS OF THE QUATERNIONIC REAL FORM !

QUANTIZATION OF $QConf(J) \Rightarrow$ MINIMAL UIR OF $QConf(J)$. GKN (2000), GP (2004-6)

WHAT ABOUT MORE GENERAL REPRESENTATIONS OBTAINED BY GEOMETRIC QUASICONFORMAL ACTIONS: THEY INCLUDE THE QUATERNIONIC DISCRETE SERIES REPRESENTATIONS OF GROSS + WALLACH DEVELOPED INDEPENDENTLY USING NON-GEOMETRIC ALGEBRAIC METHODS.
M.G, NEITZKE, PAVLYK + PIOLINE (2007)

DETAILED STUDY OF $G_{2(2)}$ AND $SU(2,1)$.

- PHYSICALLY RELEVANT REPS ARE THE MINIMAL UIR'S AND THOSE OBTAINABLE BY TENSORING OF MINREPS

c.f $AdS/_{CFT} \Rightarrow$?

THE MINREP OF A QUATERNIONIC NON-COMPACT GROUP IS IN THE CONTINUATION OF THE QUATERNIONIC DISCRETE SERIES !

HERMITIAN SYMMETRIC REAL FORM:	QUATERNIONIC SYMMETRIC REAL FORM:
$Conf(J)$ WITH MAXIMAL COMPACT SUBGROUP $\widetilde{Str_0}(J) \times U(1)$	$QConf(J)$ WITH MAXIMAL COMPACT SUBGR $\widetilde{Conf}(J) \times SU(2)$
$\Downarrow$	$\Downarrow$
HOLOMORPHIC DISCRETE SERIES	QUATERNIONIC DISCRETE SERIES
$\Downarrow$	$\Downarrow$
SINGLETONS + DOUBLETONS IN THE CONTINUATION OF HDS	MINREP IN THE CONTINUATION OF QDS
$\Downarrow$	$\Downarrow$
TENSORING SINGLETONS + DOUBLETONS YIELD THE ENTIRE HDS !	TENSORING OF MINREP YIELDS THE ENTIRE QDS ?

c.f. $AdS_5 \times S^5$, $AdS_4 \times S^7$, $AdS_7 \times S^4$
(GM 84) (GW84) (GvNW 84)

FUTURE DIRECTIONS + OPEN PROBLEMS

- CAN ONE OBTAIN THE ENTIRE QUATERNIONIC DISCRETE SERIES BY TENSORING THE MINREPS ?
- UNITARY REPRESENTATIONS OF THE DISCRETE ARITHMETIC SUBGROUPS OF U-DUALITY GROUPS ?
- QUANTIZATION OF THE QUATERNIONIC KÄHLER SIGMA MODELS IN HARMONIC SUPERSPACE ?
- GEOMETRIES OF THE MESGT'S DEFINED BY LORENTZIAN JORDAN ALGEBRAS IN FOUR AND THREE SPACE-TIME DIMENSIONS.

COMPLEX + QUATERNIONIC "ISOPARAMETRIC" SUBMANIFOLDS THAT GENERALIZE CARTAN'S REMARKABLE HYPERSURFACES.