Based on joint work with

S. Alexandrov, B. Pioline, F. Saueressig, (hep-th/0806.4620)

and earlier work with

M. Roček, ...

U. Theis

B. de Wit

D. Robles-Llana

Talk Hamburg

Linear deformations of quaternionic-Kähler manifolds

① Intro & Motivation

4n-dim. QK manifolds have holonomy groups contained in

$Sp(n) \cdot Sp(1)$

$\Downarrow$

Metrics are Einstein

$$R_{ij}(g) = \Lambda \, g_{ij}$$

"cosmological constant"

This implies

$R > 0$: compact

($R = 0$: Hyperkähler)

$R < 0$: Appears in "physics" ($N=2$ supergravity)

Typically non-compact.

Examples: non-compact versions of Wolf spaces

$$\frac{Sp(n,1)}{Sp(n) \times Sp(1)}$$

$$\frac{SO(n,4)}{SO(n) \times SO(4)}$$

$$\frac{SU(n,2)}{SU(n) \times SU(2) \times U(1)}$$

$$\vdots$$

(Alekseevsky, de Wit & Van Proeyen , ...)

Question : what are the possible deformations of a QK metric preserving the quaternionic properties ?

For $R > 0$ (complete) : there are none !

(LeBrun, LeBrun & Salamon)

For $R < 0$: deformations are possible ! (Follows from string theory !)

$\downarrow$

Mathematically not studied

This talk : study infinitesimal deformations

Method : study deformations of the twistor space of the associated Swann bundle (hyperkähler)

dim			Ex
$4n+6$	Z_S	: Twistor space	$\mathbb{R}^8 \times \mathbb{C}P^1$
$4n+4$	S	: Swann bundle	$\mathbb{R}^8$
		$\downarrow U(1) \times D$	
$4n+2$	$Z_{\mathcal{M}}$	: Twistor space	$\mathbb{C}P^3$
		$\downarrow \frac{SU(2)}{U(1)}$	
$4m$	$\mathcal{M}$	: Quaternion-Kähler	S^4

② Twistor spaces of HK

Swann bundle is Hyperkähler

$\rightarrow$ 3 HK closed 2-forms

$\omega^+ \; : \; (2,0)$

$\omega^3 \; : \; (1,1)$ Kähler - form

$\omega^- \; : \; (0,2)$

Twistor space : $(2,0)$ form w.r.t any complex structure

Ω $\quad J(\zeta,\bar\zeta)$ $\quad \zeta \in \mathbb{C}P^1$

$$\Omega = \omega^+ - i\zeta\,\omega^3 + \zeta^2\,\omega^- \qquad \text{(closed)}$$

(Hitchin, Karlhede, Lindström, Roček)
(HKLR)

Locally, $\exists$ Darboux coordinates

$$\Omega = d\mu_I \wedge d\nu^I(\varsigma)$$

(Hypermultiplets: ∞ # of auxiliary fields)

$$\mu_I(\varsigma) = w_I + \varsigma\, w_{I,1} + \cdots$$

$$\nu^I(\varsigma) = v^I_{\,} + \varsigma\, v^I_{,1} + \cdots$$

$$\Omega = \underbrace{d w_I \wedge d v^I}_{\omega^+} + i\varsigma \underbrace{\partial\bar\partial K}_{\omega^3} + \varsigma^2 \omega^- + \underline{\underline{O}}\ (\text{truncates})$$

"Solving the twistor lines" = finding Ω

$\Rightarrow$ determine $\omega_\pm$, ω_3

In general, solving the twistor lines in very difficult, unless there are extra symmetries. We will focus on

$\mathcal{M}^{4n}$: $n+1$ commuting isometries

$\Updownarrow$

$\mathcal{S}^{4(n+1)}$: $n+1$ commuting triholomorphic isometries

$\Updownarrow$

$$\nu^I(\varsigma) = v^I + x^I \varsigma - \bar{v}^I \varsigma^2 \qquad (\text{Lindstrom \& Roček})$$

($N=2$ tensor multiplet) $\quad \mathcal{O}(2)$

(*) Kähler potential

$$K(v, \bar{v}, w + \bar{w}) = \mathcal{L}(v, \bar{v}, x) - x^I (w + \bar{w})_I$$

with

$$(w + \bar{w})_I = \frac{\partial \mathcal{L}}{\partial x^I}$$

$$\mathcal{L} = \oint_C \frac{d\zeta}{2\pi i \zeta} \, H(\eta, \zeta)$$

(HKLR)

Isometries

$$w_I \to w_I + i \, \varepsilon_I$$

Solution for the twistor line

$$\mu_I(\zeta) = -\bar{\mu}_I\left(\frac{-1}{\bar\zeta}\right) + \partial_{\eta^I} H(\eta, \zeta)$$

with $\eta^I = \frac{1}{\zeta}\nu^I = \frac{v^I}{\zeta} + x^I - \bar{v}^I \zeta \qquad (*)$

<u>Ex</u>: $H = \frac{-1}{2}\eta^2$

$$\mu = w + \bar{v}\,\zeta$$

check:

$$w + \cancel{\bar{v}\,\zeta} = -\left(\bar{w} - \cancel{\frac{v}{\zeta}}\right) - \left(\cancel{\frac{v}{\zeta}} + x - \cancel{\bar{v}\,\zeta}\right)$$

$$w + \bar{w} = -x$$

$$\omega_3 = i(d\bar{w} \wedge dw + d\bar{v} \wedge dv) \qquad \mathbb{R}^4.$$

HK-geometry fixed by specifying

$H(\eta^I, \zeta)$

For Swann-bundles: indep. of ζ and

$H(\eta^I)$ homogeneous of degree one

<u>Ex</u>

$$H(\eta^I) = \frac{F(\eta^\Lambda)}{\eta^b} \qquad \eta^I = \{\eta^b, \eta^\Lambda\}$$

(Roček - Vafa - Vandoren) || c-map: F: hom. degree 2

Deformations can be studied by deforming the twistor lines.

Deformations preserving the $n+1$ commuting isometries are of the type

$$H(\eta, \zeta) = H^{0}(\eta, \zeta) + H^{1}(\eta, \zeta)$$

Ex

$$H = \frac{F(\eta^{\Lambda})}{\eta^{b}} + c\, \eta^{b} \log \eta^{b} \qquad c \in \mathbb{R}$$

(string one-lopp connection)

$$\eta^{I} \to \eta^{I} + 0$$

$$\mu_{\Lambda}^{\Lambda} \to \mu_{\Lambda}^{\Lambda} + 0$$

$$\mu_{b} \to \mu_{b} + c \ln \left[\frac{\zeta - \zeta_{+}}{\zeta - \zeta_{-}} \right]$$

$$\zeta_{\pm} = \frac{x^{b} \mp r^{b}}{2\bar{v}^{b}} \qquad r^{2} = x^{2} + 4 v \bar{v}$$

(roots of $v_{\bullet}^{b}$)

$$K = \langle \mathcal{L}eg \; \mathcal{L} \rangle_{x}$$

$K(v, \bar{v}, w + \bar{w})$ $$\mathcal{L}(v, \bar{v}, x) = \oint_{C_{+,-}} \frac{d\zeta}{2\pi i \zeta} H(\eta)$$

③ General infinitesimal deformations

Perturb twistor lines

$$\nu^I(\zeta) = \nu_0^I(\zeta) + \delta\nu^I(\zeta)$$

$$\mu_I(\zeta) = \underbrace{\overset{\circ}{\mu}_I(\zeta)}_{\substack{\text{unperturbed} \\ \text{(known)}}} + \delta\mu_I(\zeta)$$

"small"

Allow the function H to depend on

$$\rho_I \equiv \mathrm{Im}\left[\overset{\circ}{\mu}_I(\zeta)\right] = -i(w-\bar{w})_I + \cdots$$

Presence of ρ_I will break isometry

$$w_I \to w_I + i\varepsilon_I$$

$$H(\eta, \rho, \zeta) = H^{\circ}(\eta^{\circ}, \zeta) + \delta H(\eta^{\circ}, \rho, \zeta)$$

Solve for the perturbations $\delta\nu^I$, $\delta\mu_I$ in terms of unperturbed data and δH.

Solution :

$$\delta\nu^I = i \oint \frac{d\varsigma'}{2\pi i \varsigma'} \frac{\varsigma^3 + \varsigma'^3}{\varsigma'(\varsigma'-\varsigma)} \delta H^I(\eta^\circ, \varsigma, \varsigma')$$

$$\delta\mu_I = \oint \frac{d\varsigma'}{2\pi i \varsigma'} \frac{\varsigma+\varsigma'}{2(\varsigma'-\varsigma)} G_I(\eta^\circ; \varsigma, \varsigma')$$

with

$$\delta H^I = \frac{\partial}{\partial \rho_I}(\delta H)$$

$$G_I = \frac{\partial}{\partial \eta^I}(\delta H) + H^\circ_{IJ}\left(i\,\delta H^J + \frac{1}{\varsigma}\delta\nu^J\right)$$

Further (main) result : Kähler potential

$$K(u,\bar u, w, \bar w) = K_\circ(u, \bar u, w+\bar w)$$

$$+\delta K_1(u,\bar u, w-\bar w)$$

with

$$\boxed{\delta K_1 = \oint \frac{d\varsigma}{2\pi i \varsigma} \delta H(\eta, \varsigma, \cancel{\varsigma})}$$

Swann-bundles

H: hom. degree 1. in η.

(Penrose $_\wedge$ transform)
twistor

Example :

Atiyah - Hitchin metric (HK)

asymptotically ($r \to \infty$) approaches

(negative mass) Taub - Nut .

TN :

$$\mathcal{L} = -\frac{1}{2} \oint_{C_\infty} \frac{d\zeta}{2\pi i \zeta} \, \eta \ln \eta + \frac{1}{m} \oint_{C_0} \frac{d\zeta}{2\pi i \zeta} \, \eta^2$$

$$\delta H = 4 \eta \cos (4 \rho)$$

$$= 2 \eta \left(e^{4 i \rho} + e^{-4 i \rho} \right)$$

" instanton "   anti- instanton

(Coulomb - branch of
3D gauge theory)