One of the most important and beautiful resuts in Riemannian geometry is the so-called

<u>Berger</u> <u>Holonomy</u> <u>Theorem</u>. If the holonomy group of an irreducible Riemannian manifold M is not transitive on the sphere, then M is locally symmetric.

The above result follows from Marcel Berger's classification in 1955 of the holonomy groups of non locally symmetric spaces. He exploited the fact that the curvature tensor, together with its covariant derivative, takes values in the holonomy algebra.

$R_{X,Y}$, $(\nabla_X R)_{Y,Z}$

Some years later, James Simons gave a classification free proof of Berger theorem. He defined the so-called holonomy systems

$$[\mathbb{V}, R, G]$$

$\mathbb{V}$ a Euclidean vector space.

G compact and connected subgroup of $SO(\mathbb{V})$

R algebraic curvature tensor on $\mathbb{V}$ with values $R_{X,Y} \in \mathfrak{g} = Lie(G)$

The holonomy system $[\mathbb{V}, R, G]$ is called

- irreducible, if G acts irreducibly on $\mathbb{V}$.
- transitive, if G acts transitively on the sphere
- symmetric, if $g(R) = R$, for all $g \in G$

Simons proved the so-called

<u>Simons Holonomy Theorem</u>. An irreducible and non-transitive holonomy system must be symmetric.

Simons' proof, though elementary, is long and involved. It is classification free, but it makes use of case by case arguments.

Some years ago, we gave geometric proofs of both Berger and Simons theorem (Ann. of Math. 2005 and L'Enseignement Mathématique 2005)

We made use of submanifold geometry of orbits, and very in particular of the so-called normal holonomy.

We will see later a Simons' type result. The so-called Torsion Holonomy theorem or Skew-torsion Holonomy theorem. Before, we recall some results of submanifold geometry that we use for the proof

Euclidean submanifold geometry

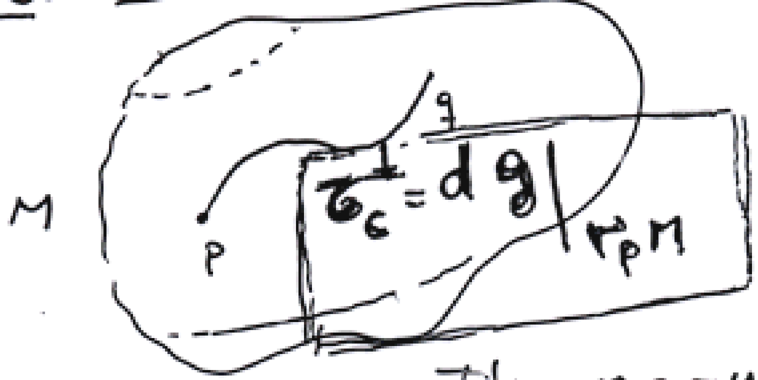
A similar rôle as symmetric spaces play in Riemannian geometry, is played by all the orbits of "s-representations" (i.e. the isotropy representation of semisimple symmetric spaces), e.g.

a symmetric space is characterized by the property that the parallel transport along a curve

$\tau_c = dg|_p$ g unique!

is achieved by the differential of an isometry.

An orbit of an s-representation is characterized by a similar property with respect to the normal parallel transport



$\tau_c^\perp = dg|_{\nu_p M}$

g isometry of $\mathbb{R}^n$
$g(M) = M$
g not unique! (in general)

INFORMAL REMARK. The normal connection gives weaker information, in submanifold geometry than the Levi-Civita connection in Riemannian geometry

Normal Holonomy Theorem (O. '90). The normal holonomy group of a Euclidean submanifolds acts on the normal space, up to its fixed point set, as an s-representation.

that is, the normal holonomy representation coincides with the holonomy representation of a symmetric space.

The above theorem is not a Berger-type theorem, since it gives information only about the representation of the normal holonomy but not about the space (i.e., the submanifold)

As we informally pointed out, normal holonomy gives weaker information than Riemannian holonomy. So, interesting applications of the normal holonomy theorem can only be given within a restrictive class of submanifolds, e.g, extrinsically homogeneous, with constant principal curvatures, complex submanifolds etc.

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For illustrating this we ennounce two Berger-type theorems for the normal holonomy

The first one is a reformulation of a well-known result of Thorbergsson about the homogeneity of isoparametric submanifolds (that can be proven using normal holonomy.

THEOREM (Thorbergsson). Let M be a submanifold of the sphere with constant principal curvatures. Assume that the normal holonomy group of M acts irreducibly and it is not transitive. Then M is an orbit of an s-representation

Theorem. (Console, Di Scala, O.) Let M be a complete and full complex submanifold of $\mathbb{C}P^n$. If the normal holonomy of M is not transitive then M is the (unique) complex orbit, in the projectivized space, of an irreducible Hermitian s-representation (or equivalently, M is an extrinsic symmetric submanifold of $\mathbb{C}P^n$).

A very important result, which is related to the normal holonomy theorem is the so-called Rank Rigidity theorem for submanifolds

Decompose the normal bundle of M as

$$\nu M = \nu_0 M \oplus \nu_0^{\perp} M$$

maximal parallel and flat subbundle

here the normal holonomy group acts as an s-representation

$$\boxed{\operatorname{rank}(M) := \dim_M (\nu_0 M)}$$

In other words, the rank is the maximal number of locally defined, linearly independent parallel normal fields

The rank rigidity theorem, in its full generality, was proven by Di Scala and the author in 2004

Let us only write the version of this result for homogeneous submanifolds.

<u>Rank Rigidity Theorem</u>. Let M^n, $n \geq 2$, be a full and irreducible homogeneous Euclidean subman.

If $\operatorname{rank}(M) \geq 2$ then M is an orbit of an (irreducible) s-representation.

<u>COROLLARY</u>. Let $M^n = K.v$ be a full and irreducible Euclidean homogeneous submanifold

Then the projection to the normal space $\nu_v M$ of any Euclidean Killing field induced by K lies in the normal holonomy algebra.

Skew-torsion Holonomy systems are

$$[\mathbb{V}, \Theta, G] \rightsquigarrow \text{Lie subgroup of } O(\mathbb{V})$$

(connected!)

(definition due to Antonio SALIERI)

$\Theta \neq 0$

$\mathbb{V}$: Euclidean vector space

Θ: 1-form with values in $\mathfrak{g} = Lie(G)$ such that $\langle \Theta_x Y, Z \rangle$ is a 3-form on $\mathbb{V}$

Such a $\mathfrak{g}$-valued 1-form arises usually as the difference tensor of two metric connections with the same geodesics (the so-called connections with skew-torsion, when one of the connections is the Levi-Civita one) We will refer to them in the last part of our talk.

Decomposition of Skew-torsion holonomy sytems

Let G' be the subgroup of G with Lie algebra

$$\mathfrak{g}' = \{ g(\Theta)_x : g \in G, x \in \mathbb{V} \}$$

Then, as it is standard to prove,

$\boxed{\mathbb{W} = \mathbb{W}_0 \oplus \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_k}$ (ortogonally)

and $\boxed{G' = G'_1 \times \cdots \times G'_k}$

where G'_i acts irreducibly on $\mathbb{W}_i$ and trivially on $\mathbb{W}_j$ $i \neq j$ (Agricola-Friedrich). Moreover,

$\leftarrow$ if $[X, \mathfrak{g}'_i] = 0$, $X \in \mathfrak{so}(\mathbb{X}_i)$, then $X = 0$

(in particular, G' is semisimple Agricola-Friedrich)

This follows from the fact that, if $\mathcal{J}$ is a complex multiplication on $\mathbb{W}_i$ (commuting with G_i) then

$\langle \Theta_{\mathcal{J}x}\mathcal{J}y, z\rangle = -\langle \Theta_x y, z\rangle$. But

$\langle \Theta_x \mathcal{J}y, \mathcal{J}z\rangle = \langle \Theta_x y, z\rangle$. Then $\Theta = 0$.

Property (*) implies that

$$\boxed{G = G_0 \times G'_1 \times \cdots \times G'_k}$$

and G_0 acts only on $\mathbb{W}_0$

No more information about G_0, since in $\mathbb{W}_0$ $g(\Theta) = 0$, for all $g \in G$.

Later we will make use of this decomposition together with the following $\rightarrow$

Skew-torsion Holonomy Theorem. Let $[\mathbb{V}, \Theta, G]$ be an irreducible non-transitive skew-torsion holonomy system. Then it must be symmetric (i.e $g(\Theta) = \Theta$ for all $g \in G$)

The proof is similar to that we gave for the Simons Holonomy theorem. (using submanifold geometry and normal holonomy, L'Enseignement Mathématique, 2005)

REMARK If $[\mathbb{V}, \Theta, G]$ is irreducible and symmetric. Then

(i) $\mathbb{V}$ is a simple Lie algebra with bracket $[x, y] = \Theta_x y$

(ii) $G = Ad(H)$, where $Lie(H) = \mathbb{V}$

(iii) Θ is unique up to scalar multiples

Naturally reductive spaces

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\, , \rangle$. The space M is said to be *naturally reductive* if there exists a reductive decomposition

$$\underbrace{\mathfrak{g}}_{Lie(G)} = \underbrace{\mathfrak{h}}_{Lie(H)} \oplus \underbrace{\mathfrak{m}}_{Ad(H)\mathfrak{m} \subset \mathfrak{m}} \qquad (*)$$

such that the geodesics by p are given by: $\gamma_{X.p}(t) = \mathrm{Exp}(tX).p$ for all $X \in \mathfrak{m}$

(where $H = G_p$)

In other words, the Riemannian geodesics of M coincides with the ∇^c-geodesic (i.e. the geodesics with respect to the canonical connection associated to $(*)$). This is in fact equivalent to the property that

$$[X, \cdot]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}, \quad \text{for all } X \in \mathfrak{m}$$

is skew-symmetric,

The Levi-Civita connection is given by:

$$\nabla_v \tilde{w} = \frac{1}{2}[\tilde{v}, \tilde{w}]_p$$

and

$$\nabla^c_v \tilde{w} = [\tilde{v}, \tilde{w}]_p$$

where, for $u \in T_pM$, $\tilde{u}$ is the Killing field on M induced by the unique $X \in m$ such that $X \cdot p = u$ (i.e., $\tilde{u}(q) = X \cdot q$)

The difference tensor of both connections is given by

$$D_v w = \nabla_v \tilde{w} - \nabla^c_v \tilde{w} = -\frac{1}{2}[\tilde{v}, \tilde{w}]_p = \boxed{-\nabla_v \tilde{w}}$$

(D is totally skew, i.e., $\langle D_v w, z\rangle$ is a 3-form)

Any isometry $g \in Iso(M)_p$ ⇝ full isotropy at p maps ∇ into itself, but, a priori, ∇^c into another canonical connection ${}^g\nabla^c$ associated to the reductive decomposition

$$\underbrace{\mathfrak{g}^g}_{Ad(g)\mathfrak{g}} = \underbrace{\mathfrak{h}^g}_{Ad(g)\mathfrak{h}} \oplus \underbrace{m^g}_{Ad(g)m}$$

We have that, if $D^g = \nabla - \nabla^g$

$$D^g_v w = -\frac{1}{2} \nabla_v \tilde{w}^g$$

the Killing field on M induced by the unique $X \in \mathfrak{m}^g$ with $X.p = w$

$$\Rightarrow \underbrace{D^g_v w - D^{g'}_v w}_{\Theta^{g,g'}_v w} = -\frac{1}{2} \nabla_v \underbrace{(\tilde{w}^g - \tilde{w}^{g'})}_{Z} = \boxed{-\frac{1}{2} \nabla_v Z}$$

$\boxed{\text{OBSERVE THAT } Z_p = 0}$

So, $(\nabla_\cdot Z)_p \in \mathrm{Lie}(\mathrm{Iso}(M)_p)$
(via the isotropy representation)

since
$$e^{t(\nabla Z)_p} = d(\varphi^Z_t)|_p$$
flux of Z.

Then
$$\Theta^{g,g'} w \doteq -\frac{1}{2} \nabla Z \in \tilde{\mathfrak{h}} = \mathrm{Lie}(\mathrm{Iso}(M)_p)$$
or
$$\Theta^{g,g'}_w \cdot = \frac{1}{2} \nabla_\cdot Z \in \tilde{\mathfrak{h}} \quad (\tilde{\Theta}^{g,g'} \text{ is totally skew})$$

that is $\boxed{\Theta^{g,g'}_w \text{ is in the full isotropy algebra}}$

$\forall\, w \in T_pM$

Let

$$\bar{h} = \{ \Theta_w^{g,g'} : w \in T_pM,\ g, g' \in Iso(M)_p \}$$

$$\bar{h} \underset{\text{ideal}}{\triangleleft} \tilde{h} = Lie\,(Iso(M)_p) \subset so(T_pM)$$

since $h.\Theta^{g,g'} = \Theta^{hg,hg'}$

Let $\bar{H}$ be the connected Lie subgroup of $SO(T_pM)$ with $Lie(\bar{H}) = \bar{h}$. Then, by what we have done for skew-torsion holonomy systems,

$$T_pM = W_0 \oplus W_1 \oplus \cdots \oplus W_k \quad \text{(orthogonally)} \qquad *$$

and

$$\underset{\substack{\text{connected}\\\text{component}\\\text{of id}}}{Iso_0(M)_p} = H_0 \times \bar{H} = H_0 \times \bar{H}_1 \times \ldots \times \bar{H}_k$$

and

$\boxed{(i)}$ $\bar{H}_i$ acts irreducibly on W_i and trivially on W_j if $i \neq j$

$\boxed{(ii)}$ Either $\bar{H}_i$ is transitive on the sphere of W_i or $\bar{H}_i$ acts as the Ad-reprentation of a compact simple Lie group into its Lie algebra (i.e. as the isotropy representation of a Lie group with the biinvariant metric $G \times G / diag(G \times G)$)

$\boxed{(iii)}$ No element $x \neq 0$ in $so(W_i)$ commutes with $\bar{h}_i = Lie(\bar{H}_i)$

$\boxed{(iv)}$ H_0 acts only on W_0 = fixed set of $\bar{H}$ (in general, with no properties)

Theorem (O.-Reggiani) Let $M = G/G_p$ be a locally irreducible homogeneous Riemannian manifold such that $(G_p)_0$ can be written as before (with properties (i) to (vi)). Then either $H_0 = \{id\}$ or $\bar{H} = \{id\}$ (In the first case only for at most one i $\bar{H}_i \neq \{id\}$)

Proof (sketched)

The main difficulty is when the isotropy representation of $(G_p)_0$ has fixed non-trivial vectors

Choose $i > 0$, let us assume $i = 1$

Define the following subspaces of T_pM.

$\mathcal{D}_p$ = fixed vectors of $H_0 \times H_2 \times \dots H_k$. (via isotropy rep)

$\mathcal{D}'_p$ = fixed vectors of H_1 (")

Then $\mathcal{D}_p = \underbrace{\mathbb{W}}_{\text{fixed vectors of } (G_p)_0} \oplus \mathbb{W}_1$

and $\mathcal{D}'_p = \mathbb{W}_1^\perp$

$(\mathcal{D}_p \cap \mathcal{D}'_p = \mathbb{W})$

Both $\mathcal{D}_p$ and $\mathcal{D}'_p$ extend to autoparallel distributions of M (locally)

A vector $w \in \mathbb{W}$ extends to a G-invariant vector field $\tilde{w}$. This field has to be "divergence free" when restricted to any integral manifold $S(x)$ of of $\mathcal{D}$.

$(\nabla\tilde{\omega})_p$ commutes with H_1. So

$\underbrace{(\nabla\tilde{\omega})_p^{sym}}_{\lambda\, Id}$, $\underbrace{(\nabla\tilde{\omega})_p^{skew}}$ commutes with H_1

must be zero on W_1 by property (iii)

(when restricted to W_1)

but $\lambda = 0$ since

$\tilde{\omega}$ is divergence free in $S(p)$

This implies that W extends to a parallel distribution on $S(p)$. Then parallel distribution of W_1 gives rise to a parallel distribution of $S(p)$. This shows that W_1 gives rise to an <u>autoparallel</u> distribution on M which is complementary to $\mathcal{S}$. □

Then M splits.

COROLLARY Let $M = G/H$ be a compact naturally reductive homogeneous Riemannian manifold. Assume that M is locally irreducible and that it is not a globally rankone symmetric space. Then

(i) $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(M, \nabla^c)$

(ii) $\mathrm{Iso}(M) \subset \mathrm{Aff}(M, \nabla^c)$, unless M is a Lie group with the biinvariant metric (in that case the symmetry moves ∇^c to its opposite with respect to ∇ (i.e. $\sigma_*(\nabla^c) = 2\nabla - \nabla^c$)

This generalizes several partial results (for $K>0$, Grove-Shankar-Ziller) or for G-simple (Onischik). All of them case by case and using tables. In particular answer a question of Wolf in his classification of (strongly) isotropy irreducible spaces: why the isotropy of such spaces cannot be extended, except for rank one-symmetric spaces.