Ricci Flow Unstable Cell Centered at a Kähler-Einstein Metric on the Twistor Space of Positive Quaternion Kähler Manifolds

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Abstract. We propose a notion of "Ricci flow unstable cell" which extends Einstein metrics. We hope that once we have a "Ricci flow unstable cell" centered at an Einstein metric, we can extract more geometric information by analyzing the corresponding Ricci flow ancient solution. As an example of this idea, we construct a "Ricci flow unstable cell" centered at a Kähler-Einstein metric on the twistor space of positive quaternion Kähler manifolds. By analyzing the corresponding ancient solutions, we settle the LeBrun-Salamon conjecture, i.e., we prove that any locally irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces. Details can be found in [K-O1,2] arXiv:0801.2605, 0805.1956 [math.DG].

0. Background.

Let M be an n-dimensional smooth closed manifold. **Perelman's $\mathcal{W}$-functional** is defined by

$$\mathcal{W}^m(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] dm$$

where $dm = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$. We put the constraint that the measure dm is a fixed volume form on M. The L^2-gradient flow of the functional $\mathcal{W}^m$ under this constraint is

$$\begin{cases} \partial_t g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f) \ , \\ \partial_t f = -\triangle f - R + \dfrac{n}{2\tau} \ , \\ \partial_t \tau = -1 \ . \end{cases} \tag{1}$$

The difficulty with this system of equations is that there is no guarantee that the solution exists even for a short time (the second equation is "backward" and the first and the second equations are coupled). However, this difficulty disappears if we introduce the following modification of the above equations:

$$\begin{cases} \partial_t g_{ij} = -2R_{ij} \ , \\ \partial_t u = -\triangle u + Ru \qquad (u := (4\pi\tau)^{-\frac{n}{2}} e^{-f}) \ , \\ \partial_t \tau = -1 \ . \end{cases} \tag{2}$$

In this system, the first equation is the Ricci flow where the short time existence is established after the works by by Hamilton and DeTurck. Therefore, the second equation (conjugate heat equation) is solved in the backward direction with the "initial" condition in the future time. The relationship between (1) and (2) is this:

Apply the 1-parameter family of time-dependent diffeomorphisms generated by the time-dependent vector field $-\nabla f$ to (2). Then we get (1). Now the advantage of (2) is that the functional

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$$

is monotone nondecreasing along the solution of (2). Indeed, we have the "entropy formula" ([P])

$$\frac{d}{dt}\mathcal{W} = 2\int_M \tau \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 u dV \geq 0 \ .$$

Here, in the case of (1) udV should be replaced by dm. Perelman's $\mathcal{W}$-functional is a "coupling" of the **logarithmic Sobolev functional**[1] and the **Hilbert-Einstein functional**[2]. Suppose that there exists a critical point which corresponds to a Ricci soliton

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0$$

which at time $t = -1$ ($\tau = 1$) is interpreted as the initial condition for the Ricci flow equation (the solution satisfies the above equation and called the Ricci soliton, which evolves under a 1-parameter group of diffeomorphisms of M). Perelman [P] showed that this Ricci soliton is characterized by the equality case of the logarithmic Sobolev inequality in the following way. Let $g_{ij}(-1)$ satisfy the above equation at time $t = -1$ and $g_{ij}(t)$ the corresponding solution of the Ricci flow, i.e., the Ricci soliton with initial metric $g_{ij}(-1)$. Then the logarithmic Sobolev inequality on $(M, g_{ij}(t))$ introduced in [P] is

$$\begin{aligned} W(g_{ij}(t)\widetilde{f}, -t) &\geq W(g_{ij}(t), f(t), -t) \\ &= \inf_{\widetilde{f}: \int_M (4\pi(-t))^{\frac{n}{2}} e^{-\widetilde{f}} dV_{g(t)} = 1} W(g_{ij}(t), \widetilde{f}, -t) \\ &=: \mu(g_{ij}(t), -t) \\ &= \mu(g_{ij}(-1), 1) \end{aligned}$$

where $\widetilde{f}$ is any smooth function on M satisfying the condition

$$\int_M (4\pi(-t))^{\frac{n}{2}} e^{-\widetilde{f}} dV_{g(t)} = 1 \ .$$

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[1] The logarithmic Sobolev inequality on the n-dimensional Euclidean space $\mathbb{R}^n$ is the following. Let $f = f(x)$ satisfies the constraint $\int_{\mathbb{R}^n} (4\pi\tau)^{-\frac{2}{n}} e^{-f} dV_{\mathrm{euc}} = 1$. Then we have

$$\int_{\mathbb{R}^n} [\tau |\nabla f|^2 + f - n](4\pi\tau)^{-\frac{2}{n}} e^{-f} dV_{\mathrm{euc}} \geq 0$$

where the equality holds iff $f(x) = \frac{|x|^2}{4\tau}$.

[2] The Hilbert-Einstein functional is $\displaystyle\int_M R dV_g$ for a closed Riemannian manifold (M, g) and the critical points are Einstein metrics.

This observation gives us an important information on the behavior of the W-functional at a critical point (i.e., the Ricci soliton). We look at the Hessian of the $\mathcal{W}^m$-functional at the critical point. The $\mathcal{W}^m$-functional is invariant under the group of all dm-preserving diffeomorphisms and therefore this action corresponds to the zeros of the Hessian. On the other hand, the action of the diffeomorphisms which do not preserve dm may be given by the following way. Let ϕ be such a diffeomorphism. Introduce f^ϕ by setting $dm = (4\pi\tau)^{-\frac{2}{n}} e^{-f^\phi} dV_{\phi^* g}$ and define $\phi^*(g, f, \tau) = (\phi^* g, f^\phi, \tau)$. Then we have

$$\mathcal{W}^m(\phi^*(g, f, \tau)) = \int_M [\tau(R_{\phi^* g} + |\nabla f^\phi|^2_{\phi^* g}) + f^\phi - n] \underbrace{(4\pi\tau)^{-\frac{n}{2}} e^{-f^\phi} dV_{\phi^* g}}_{dm}$$

and therefore the $\mathcal{W}^m$-functional increases in the direction of the action of the diffeomorphisms which do not preserve dm, which follows from the logarithmic Sobolev characterization of the Ricci soliton. This implies that the tangent space of the configuration space $\{(g, f, \tau)\}$ decomposes into three subspaces V_0, V_+ and V_-. Here, V_0 corresponds to the action of the dm-preserving doffeomorphisms (Hess $= 0$), V_+ corresponds to the action of the diffeomorphisms which do not preserve dm (Hess > 0) and finally V_- corresponds to the rest[3].

Applications of the $\mathcal{W}$-functional.

1. No Local Collapsing Theorem (Perelman). If the Ricci flow $\partial_t g_{ij} = -2R_{ij}$ defined on $[0, T)$, then $\exists\, \kappa := \kappa(g_{ij}(0), T) > 0$ such that $(M, g_{ij}(t))$ is κ-non collapsing in scale $\sqrt{T}$ (i.e., $\forall r < \sqrt{T}$, $|\mathrm{Rm}|(x) \le r^{-2}$ $\forall x \in B(r) \Rightarrow \mathrm{Vol}(B(r)) \ge \kappa r^n$).

One of the important consequences of No Local Collapsing Theorem is that if a singularity develops in the Ricci flow in finite time, then an appropriate rescaling procedure produces an **ancient solution** which encodes all information of the singularity. Here, a Ricci flow solution is called an ancient solution if it is defined in the time-interval $(-\infty, T)$, T being a real number.

2. Dynamical Stability of a Positive Kähler-Einstein Metric under the Kähler-Ricci Flow (Perelman, Tian-Zhu [T-Z]). If a Fano manifold M admits a Kähler-Einstein metric, then the normalized Kähler-Ricci flow with any initial metric in $c_1(M)$ converges to a Kähler-Einstein metric in the sense of Gromov-Cheeger.

Therefore the Kähler-Ricci flow produces a Ricci flow stable cell centered at a positive Kähler-Einstein metric. It is natural to search for an example of a Ricci flow unstable cell centered at a Kähler-Einstein metric on a Fano manifold. Such unstable cell, if exists, consists of *ancient* solutions of *non-Kähler* Ricci flow. In this paper we propose a candidate for such possibility. The ancient solution proposed in this paper corresponds to one of the natural collapses of the twistor space of positive quaternion Kähler manifolds[4], in which the base manifold (= a given positive quaternion Kähler manifold) shrinks faster.

[3] This is very similar to the behavior of the Hilbert-Einstein functional under the Yamabe problem.

[4] There are two kinds of natural collapses of the twistor fibration $Z \to M$ of a positive quaternion Kähler manifold. One may ask which shrinks faster, base manifold or a fiber.

1. Main results.

Let (M^{4n}, g) be any locally irreducible positive quaternion Kähler manifold of dimension $4n \geq 8$. The local holonomy group is contained in $\mathrm{Sp}(1)\mathrm{Sp}(n)$ and we can consider the holonomy reduction $\mathcal{P} \to M$ of the oriented orthonormal frame bundle of M. Write $\mathcal{Z} \to M$ for the twistor fibration and let

$$\underbrace{\alpha_1, \alpha_3}_{\text{fiber } \mathbb{P}^1 \text{ direction}} \quad , \quad \underbrace{X^i \ (i = 0, 1, 2, 3)}_{\text{base } M \text{ direction}}$$

denote the unitary (moving) coframe on $\mathcal{Z}$. This set-up is not in the complex Kähler setting but in the real Riemannian setting w.r.to the Kähler-Einstein metric on $\mathcal{Z}$. Here, the triple $\{\alpha_i\}_{i=1}^3$ constitutes the $\mathrm{Sp}(1)$-part of the connection form defined on the holonomy reduction $\mathcal{P}$ of the oriented orthnormal frames of the positive quaternion Kähler manifold (M, g). We take α_1, α_3 from the triple $\{\alpha_i\}_{i=1}^3$. This choice correspond to looking at the infinitesimal variation of orthogonal complex structures around the orthogonal complex structure J of a tangent space of M represented by $(0 : 1 : 0)$ in the $\mathbb{P}^1$-fiber of the twistor fibration $\mathcal{Z} \to M$. The quadruple $\{X^i\}_{i=0}^3$ consists of column n-vectors corresponds to the decomposition of the orthogonal complex structure J and the quaternion structure, which constitute an orthonormal coframe of M defined on $\mathcal{P}$.

Using the above data, we introduce the following two parameter family of Riemannian metrics on $\mathcal{Z}$:

$$\mathcal{F} = \{\rho g_\lambda^{\mathrm{CY}}\}_{\rho, \lambda > 0}$$

where $\rho g_\lambda^{\mathrm{CY}}$ is a Riemannian metric on $\mathcal{Z}$ defined in terms of the Cartan formalism of the moving frames by

$$\begin{aligned} \rho\, g_\lambda^{\mathrm{CY}} := \rho\ \{\ \lambda^2\ (& \underbrace{\alpha_1^2 + \alpha_3^2}_{\text{Fubini-Study metric on } \mathbb{P}^1\text{-fiber}}) \\ & + \underbrace{{}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3}_{\text{quaternion Käler metric on the base manifold } M} \} \ . \end{aligned}$$

Proposition 1.1 (Chow-Yang [C-Y]). *The metric $\rho g_\lambda^{\mathrm{CY}}$ is Kähler if and only if $\lambda = 1$ (indeed, g_1 is Kähler-Einstein).*

Theorem 1.2 (Theorem 7.1). *(1) For the above family*

$$\mathcal{F} = \{\rho g_\lambda^{\mathrm{CY}}\}_{\rho, \lambda > 0}$$

of Riemannian metrics on the twistor space $\mathcal{Z}$ of a quaternion Kähler manifold M^{4n}, we have the formula

$$\mathrm{Ric}_\lambda = 2\ \lambda^{-2}\, \{1 + (2n+1)\, \lambda^2\}\ g_{\sqrt{\frac{2\lambda^2(1+n\lambda^2)}{1+(2n+1)\lambda^2}}} \ .$$

In particular $\rho g_\lambda^{\mathrm{CY}}$ is Kähler-Einstein if and only if $\lambda = 1$.

(2) Any Ricci flow solution with initial metric in $\mathcal{F}$ stays in $\mathcal{F}$ and is an ancient solution whose asymptotic soliton is the special solution consisting of positive multiples of the Kähler-Einstein metric g_1. In particular the 2-parameter family $\mathcal{F} \cong \{(\lambda, \rho)\}_{\rho,\lambda>0}$ is foliated by the trajectories of the Ricci flow solutions which are given by the equation

$$\rho = (\mathrm{const})\, \frac{\lambda^2}{|1-\lambda^2|^{2n+2}}$$

where the (const) *is positive and depends on the initial metric.*

Example 1.3. Pick a trajectory defined by the equation

$$\rho = \frac{c\lambda^2}{(\lambda^2-1)^{2(n+1)}}$$

where $c > 0$ and $\lambda > 1$ in the (λ, ρ)-plane identified with the family $\mathcal{F}$. This trajectory consists of metrics $\rho g_\lambda^{\mathrm{CY}} = \rho[\lambda^2(\alpha_1^2 + \alpha_3^2) + \sum_{i=0}^3 {}^t X^i \cdot X^i]$ with $\rho = \frac{c\lambda^2}{(\lambda^2-1)^{2(n+1)}}$. As $\mathrm{Ric}_\lambda = 2\lambda^{-2}\{1+(2n+1)\lambda^2\}[\frac{2\lambda^2(1+n\lambda^2)}{1+(2n+1)\lambda^2}(\alpha_1^2+\alpha_3^2) + \sum_{i=0}^3 {}^t X^i \cdot X^i]$, we have

$$\mathrm{Scal}(\rho g_\lambda^{\mathrm{CY}}) = \frac{8(1+n)(\lambda^2-1)^{2(n+1)}(1+2n\lambda^2)}{c\lambda^4}$$

for the scalar curvature of the metric $\rho g_\lambda^{\mathrm{CY}}$ in the trajectory. If we set $u =$ constant determined by $\int_M u dV = 1$, i.e., $u = 1/\mathrm{Vol}(g_{ij}(t))$, we get a solution $u(t,x)$ (t-dependent constant function on M) to the conjugate heat equation $\partial_t u = -\triangle u + Ru$. Since $\mathrm{Vol}(\rho g_\lambda^{\mathrm{CY}}) = (\mathrm{Vol}(M,g))\rho^{2n+1}\lambda^2$, we have

$$u = \frac{(\lambda^2-1)^{2(n+1)(2n+1)}}{\lambda^{4(n+1)}\mathrm{Vol}(M,g)} .$$

From the Ricci flow equation (Theorem 7.1 (1)) we have

$$\frac{d(c\lambda^4/(\lambda^2-1)^{2n+2}))}{-8(1+n\lambda^2)} = dt .$$

Therefore if we set

$$\tau = \int_\infty^\lambda \frac{d/dl(cl^4/(l^2-1)^{2(n+1)})}{8(1+nl^2)} dl ,$$

then the function $-W(g_{ij}, f, \tau)$ (W being Perelman's W-functional) is monotone decreasing along the Ricci flow trajectory passing through a metric $\rho g_\lambda^{\mathrm{CY}}$ with $\lambda > 1$, which is determined by the triple $(\rho g_\lambda^{\mathrm{CY}}, f, \tau)$ where ρ, λ, τ are given as above, $\lambda \in (1, \infty)$ increases to from 1 to ∞ when τ decreases from ∞ to 0), and f is determined by setting $u = (4\pi\tau)^{-(2n+1)}e^{-f}$ with u and τ given as above.

Theorem 1.4 (Theorem 7.2). *For any locally irreducible positive quaternion Kähler manifold, the limit formula*

$$\lim_{\lambda\to\infty} |\nabla^{g_\lambda^{\mathrm{CY}}} \mathrm{Rm}^{g_\lambda^{\mathrm{CY}}}|_{g_\lambda^{\mathrm{CY}}} = 0$$

holds.

By applying of this limit formula, we settle the LeBrun-Salamon conjecture :

Theorem 1.5 (Theorem 7.3). *Any locally irreducible positive quaternion Kähler manifold is isometric to one of the Wolf spaces, i.e., the formula*

$$|\nabla \mathrm{Rm}| = 0$$

holds.

The proofs with full details of all theorems in this section can be found in [K-O1], arXiv:0801.2605 [math.DG].

2. Quaternion Kähler manifolds.

Let $\mathbb{H}$ denote the quaternions and identify $\mathbb{R}^{4n} = \mathbb{H}^n$. Then $\mathbb{H}$ acts on $\mathbb{H}^n$ from the right which makes $\mathbb{R}^{4n} = \mathbb{H}^n$ into a right $\mathbb{H}$-module. Define

$$\mathrm{Sp}(n) = \{A \in SO(4n) \,|\, A \text{ is } \mathbb{H}\text{-linear}\} \ .$$

Let $\mathrm{Sp}(1)$ be the subgroup of $\mathrm{SO}(4n)$ consisting of the image in $\mathrm{SO}(4n)$ of the right action of the group of unit quaternions on $\mathbb{H}^n$. Then we can define the subgroup $\mathrm{Sp}(n)\mathrm{Sp}(1)$ of $\mathrm{SO}(4n)$ to be the product of the subgroups $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ in $\mathrm{SO}(4n)$. This is a proper subgroup if $n \geq 2$.

Definition 2.1. A $4n$ ($n \geq 2$)-dimensional Riemannian manifold is **quaternion Kähler**, if its holonomy group lien is $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

Throughout this paper we restrict our attention to locally irreducible (in the sense of the de Rham decomposition) positive quaternion Kähler manifolds.

• The locally irreducible quaternion Kähler condition implies the Einstein condition. Therefore quaternion Kähler manifolds are classified into three classes according to the sign of the scalar curvature :

• A (geodesically) complete quaternion Kähler manifold is called **positive** (resp. loc. hyperKähler, negative), if its scalar curvature is positive (resp. zero, negative).

• loc. hyperKähler $\Leftrightarrow$ No $\mathrm{Sp}(1)$ component.

• A positive quaternion Kähler manifold is a simply connected positive Einstein manifold.

Normalization. We fix the scale of the invariant metric of $\mathbb{HP}^n$ so that the sectional curvatures range in the interval $[1, 4]$. This is equivalent to saying $\mathrm{Ric}(g_{\mathbb{HP}^n}) = 4(n+2)g_{\mathbb{HP}^n}$ and therefore to the statement $\mathrm{Scal}(g_{\mathbb{HP}^n}) = 16n(n+2)$. Set $\widetilde{S} := 16n(n+2)$ (in this paper we normalize a positive quaternion Kähler metric so that its scalar curvature is equal to $\widetilde{S}$). We fix the scale of the Fubini-Study metric of the $\mathbb{P}^1$-fiber of the twistor fibration and other cases so that the Gaussian curvature is identically 4.

Definirion 2.2. *A quaternion Kähler manifold in dimension 4 is defined as a self-dual Einstein Riemannian 4-manifold.*

In this paper we consider only positive quaternion Kähler manifolds of dimension ≥ 8. The round 4-sphere S^4 and the complex projective space $\mathbb{P}^2(\mathbb{C})$ with the Fubini-Study metric exhaust examples of positive quaternion Kähler 4-manifolds (Hitchin [Hi] and Friedrich-Kurke [F-K]). The application of the methods of this paper gives us a new proof to this result ([K-O2]).

3. Examples of quaternion Kähler manifolds of dimension ≥ 8.

Example 3.1. (1) [Positive quaternion Kähler manifolds] **Wolf spaces** (positive quaternion Kähler symmetric spaces)

$$\mathbb{P}^n(\mathbb{H}) = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)} \ ,$$
$$\mathrm{Gr}_2(\mathbb{C}^n) = \frac{\mathrm{SU}(n)}{S(U(n-2) \times U(2))} \ ,$$
$$\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n) = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times SO(4)}$$

plus some exceptional cases.

These spaces are compact symmetric spaces whose isotropy group contains an Sp(1)-component.

(2) [Negative quaternion Kähler manifolds] The non-compact dual of Wolf spaces are examples of negative quaternion Kähler manifolds. There exist many other examples of noncompact negative quaternion Kähler manifolds which are <u>not</u> symmetric (e.g. Alexeevskii, Galicki, $\cdots$).

Remark 3.2. Galicki-Lawson's quaternion Kähler reduction method produces many examples of positive quaternion Kähler orbifolds which are <u>not</u> symmetric.

In Theorem 7.3 of this paper, we give an affirmative answer to the following conjecture :

Conjecture 3.3 (LeBrun-Salamon). Any positive quaternion Kähler manifold is a Wolf space.

4. Moving frames.

<u>Basic Setting</u> :

- (M^{4n}, g) : a quaternion Kähler manifold

$\Rightarrow \exists$ reduction of the SO$(4n)$ frame bundle $\mathcal{F}$ to the holonomy Sp(n)Sp(1) bundle $\mathcal{P}$ (i.e., we fix an orthonormal frame in $\mathcal{F}$ at one point and think of all parallel displacements to any point along various curves $\Rightarrow$ we get $\mathcal{P}$). Each point of $\mathcal{P}$ over $m \in M$ represents an orthonormal frame at $m \in M$ tautologically. $\Rightarrow$ The space $\mathcal{P}$ is the domain where the orthonormal frames obtained by all parallel displacements are defined simultaneously.

• Definition of subgroups $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ of $\mathrm{SO}(4n)$ $\Rightarrow$ $\mathrm{Sp}(n)$ is the centralizer of $\mathrm{Sp}(1)$ in $\mathrm{SO}(4n)$ $\Rightarrow$ $(e_A)_{A=1}^{4n} \in \mathcal{P}$: an orthonormal frame at $m \in M$ gives an identification $T_mM \to \mathbb{H}^n$ defined by

$$x^A e_A \mapsto (x^a + ix^{n+a} + jx^{2n+a} + kx^{3n+a})_{a=1}^n$$

and a local section $(e_A)_{A=1}^{4n}$ of $\mathcal{P} \to M$ on an open set $U \subset M$ defines a right $\mathbb{H}$-module structure on TU $\Rightarrow$ locally defined three almost complex structures I, J, K (behaving like i, j, k in $\mathbb{H}$) which is not parallel if the $\mathrm{Sp}(1)$ part of the holonomy is non-trivial (we are interested in this case).

Linear Algebra :

• The right action of i and j on $\mathbb{R}^{4n} = \mathbb{H}^n$:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

• The Lie algebra $\mathrm{sp}(n)$ is computed as

$$\begin{pmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{pmatrix}$$

where $A_0 = -{}^tA_0$ and $A_\lambda = {}^tA_\lambda$ $(1 \le \lambda \le 4)$ are $n \times n$ matrices. Similarly, the Lie algebra of the subgroup $\mathrm{Sp}(1)$ of $\mathrm{SO}(4n)$ is computed as

$$\begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & a_3 & -a_2 \\ a_2 & -a_3 & 0 & a_1 \\ a_3 & a_2 & -a_1 & 0 \end{pmatrix}$$

$\Rightarrow$ The Lie algebra of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is

$$\begin{pmatrix} A_0 & -A_1 - a_1 & -A_2 - a_2 & -A_3 - a_3 \\ A_1 + a_1 & A_0 & -A_3 + a_3 & A_2 - a_2 \\ A_2 + a_2 & A_3 - a_3 & A_0 & -A_1 + a_1 \\ A_3 + a_3 & -A_2 + a_2 & A_1 - a_1 & A_0 \end{pmatrix}. \tag{3}$$

In the following computation we use α_i instead of a_i $(i = 1, 2, 3)$ for $\mathrm{sp}(1)$-valued 1-forms.

Cartan Formalism :

• (M^{4n}, g) : a quaternion Kähler manifold $(n \ge 2)$. $\mathcal{P}$: the holonomy reduction of the full frame bundle. $(\theta^A)_{A=1}^{4n}$: the orthonormal coframe dual to the orthonormal frame $(e_A)_{A=1}^{4n}$ in $\mathcal{P}$. $(\theta_A)_{A=1}^{4n}$ is a system of 1-forms on $\mathcal{P}$. The geometry of M being encoded in $(d\theta_A)_{A=1}^{4n}$ is the main idea of the Cartan formalism.

• The 1-st and 2-nd structure equations

$$\begin{aligned} d\theta^A + \Gamma^A_B \wedge \theta^B &= 0 \\ d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B &= \Omega^A_B \end{aligned}$$

where (Γ^A_B) represents the Levi-Civita connection matrix (i.e., $\mathrm{sp}(n) + \mathrm{sp}(1)$-valued 1-form on $\mathcal{P}$) and (Ω^A_B) is the curvature matrix (i.e., $\mathrm{sp}(n) + \mathrm{sp}(1)$-valued 2-form on $\mathcal{P}$) both of which is of the form (3).

• meaning of the connection and curvature matrices : $(e_A)^{4n}_{A=1} \in \mathcal{P}$: an orthonormal frame at $T_m M \Rightarrow$

$$\begin{aligned} \nabla\,(e_1, \dots, e_{4n}) \,&=\, (e_1, \dots, e_{4n})\,(\Gamma^A_B)\ , \\ R(X,Y)\,(e_1, \dots, e_{4n}) \,&=\, (e_1, \dots, e_{4n})\,(\Omega^A_B(X,Y))\ . \end{aligned}$$

5. Twistor spaces. The Chow-Yang metrics.

Definition of the Twistor Space :

• There is a canonical identification

$$\{\text{unit pure imaginary quaternions}\} \xleftrightarrow{\text{right action of unit pure imaginary quaternions}} \{\text{orthogonal complex structures on } TM_m\}\ .$$

• The above identification depends on the basis $(e_A) \in \mathcal{P}$. However, if q is a unit pure imaginary quaternion, then so is xqx^{-1} for any unit quaternion x and therefore the set (identified with $\mathbb{P}^1$) of all orthogonal complex structures on TM_m is independent of the choice of the basis $(e_A) \in \mathcal{P}$. The twistor space $\mathcal{Z}$ of M is defined by

$$\mathcal{Z} = \mathcal{P} \times_{\mathrm{Sp}(n)\mathrm{Sp}(1)} \mathbb{P}^1$$

where $\mathrm{Sp}(n)\mathrm{Sp}(1)$ operates on the set $\mathbb{P}^1$ of unit pure imaginary quaternions (orthogonal complex structures of $\mathbb{H}^n$) by the trivial action of $\mathrm{Sp}(n)$ and the right action of the group $\mathrm{Sp}(1)$ of unit quaternions given by $q \mapsto xqx^{-1}$. From this we have

$$\mathcal{Z} \,=\, \mathcal{P}\,/\,\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)\ .$$

Almost Complex Structure and Chow-Yang Metrics on $\mathcal{Z}$:

• The Levi-Civita connection of (M^{4n}, g) corresponds to the horizontal distribution on the holonomy reduction $\mathcal{P} \to M$ of the oriented orthonormal frame bundle. The twistor space $\mathcal{Z}$ is by definition $\mathcal{Z} = \mathcal{P}/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ and therefore we can introduce canonically the horizontal distribution and submersion metric on $\mathcal{Z}$ from those of $\mathcal{P}$. The almost complex structure of $\mathcal{Z}$ is defined by the corresponding orthogonal complex structure in the horizontal subspace and the standard complex structure of $\mathbb{P}^1$ along the fiber.

Theorem 5.1 (Salamon 1982). *The orthogonal almost complex structure on the twistor space is integrable.*

• We introduce a certain class of Riemannian metrics on $\mathcal{Z}$ and compute the Ricci tensor by moving frame technique (we will call this the class of Chow-Yang metrics)[5]. The construction of this class of metrics is conceptually not so simple and therefore we give a detailed description before starting moving frame computations. We start by recalling the idea of the Cartan formalism of moving frames. Let (N, g) be any n-dimensional oriented Riemannian manifold and $\mathcal{F} \to N$ the bundle of all oriented orthonormal frames. We have the system $\{\theta^1, \ldots, \theta^n\}$ of coframes on $\mathcal{F}$ which is, at $p \in \mathcal{F}$ lying over $m \in N$, the system of 1-forms dual to the orthonormal frame of N_m represented by the point $p \in \mathcal{F}$. Given a local frame field on an open set $U \subset N$, we tautologically associate the section $U \to \mathcal{F}$. Thus the local frames which are not unique on N becomes a globally defined single valued object on $\mathcal{F}$ and moreover the dual object $\{\theta_1, \ldots, \theta_n\}$ consists of differential 1-forms and therefore we have an advantage being able to work functorially on differential forms (such as connection forms) on $\mathcal{F}$. For instance, the Riemannian metric on N is written as $(\theta^1)^2 + \cdots + (\theta^n)^2$ and connection form is computed by taking the exterior differential of $\{\theta_1, \ldots, \theta_n\}$ on $\mathcal{F}$ and so on.

Now let us return to our original (quaternion Kähler) situation. A fiber on $m \in M$ of the twistor fibration $\mathcal{Z} \to M$ is the set of all orthogonal complex structures on the tangent space M_m which is canonically identified with $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \cong \mathbb{P}^1$. Therefore the twistor space is also defined as the orbit space with respect to the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$ action on $\mathcal{P}$, i.e.,

$$\mathcal{Z} = \mathcal{P}/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \ .$$

We construct local sections $\mathcal{Z} \to \mathcal{P}$ of the principal $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-bundle $\mathcal{P} \to \mathcal{Z}$ in the following way (we use these local sections to construct a certain class of metrics on $\mathcal{Z}$). Fix a point $m \in M$. Let $\mathbb{P}^1_m \subset \mathcal{Z}$ be the fiber of the twistor fibration over m. To each $z \in \mathbb{P}^1_m$ we (locally) associate a quaternion orthonormal frame in the fiber of $\mathcal{P} \to M$ over m so that the frame is ordered in the way compatible with respect to the orthogonal complex structure represented by z. If z varies on $\mathbb{P}^1_m$ such frames rotates by an element of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and the rotation is unique modulo those by elements of $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$. This procedure is possible only locally on $\mathbb{P}^1_m$ because this is equivalent to make the (local) section of the principal $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-bundle $\mathrm{Sp}(n)\mathrm{Sp}(n) \to \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{U}(2n) \cong \mathbb{P}^1$. We extend this construction locally on open set $U \subset M$ containing m. This way we construct local sections of the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-principal bundle $\mathcal{P} \to \mathcal{Z}$. We then restrict 1-forms

$$X^0, X^1, X^2, X^3$$

[5] The class of Chow-Yang metrics is not identical to the so called canonical deformation on the twistor space $\mathcal{Z}$, where the canonical deformation consists of metrics constructed by the sum of the base metric and scaled fiber metric by using the horizontal distribution. The following discussion shows that the construction of the Chow-Yang metric is different from the canonical deformation at least from topological nature.

and the

> Sp(1)-part of the connection form orthogonal to the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-fiber which are defined globally on $\mathcal{P}$ (these are contained in the space of 1-forms spanned by α_i's)

to the above constructed local sections. The Sp(1)-part of the connection form restricted to the local sections are not necessarily unit 1-forms with respect to the standard submersion metric on Z coming from the standard Riemannian submersion $\mathcal{P} \to \mathcal{Z}$ (because the local sections are not orthogonal to the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-fibers)[6]. We thus get the system of 1-forms

$$\{X^0, X^1, X^2, X^3, \alpha_1, \alpha_3\}$$

say (this corresponds to the infinitesimal deformation of orthogonal complex structures at the one defined by the right multiplication of j, as in the following computations), locally at 1 point on Z. We define the metric on Z by requiring that the above constructed system of 1-forms to be an orthnormal coframe. This means that the metric

$$g_1^{\mathrm{CY}} := (\alpha_1^2 + \alpha_3^2) + {}^tX^0X^0 + {}^tX^1X^1 + {}^tX^2X^2 + {}^tX^3X^3$$

is an expression in terms of the orthnormal coframes. In the following arguments we will consider the metrics of the form

$$g_\lambda^{\mathrm{CY}} := \lambda^2(\alpha_1^2 + \alpha_3^2) + {}^tX^0X^0 + {}^tX^1X^1 + {}^tX^2X^2 + {}^tX^3X^3$$

on $\mathcal{Z}$ (we call this type of metric as a Chow-Yang metric, because Chow and Yang first constructed such metrics in [C-Y]). These metrics are well-defined (independent of the choice of the local sections of the $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-bundle $\mathcal{P} \to \mathcal{Z}$) and moreover we can work functorially on differential forms on $\mathcal{P}$ using the moving frame technique.

- We introduce the so called canonical deformation metric

$$g_\lambda^{\mathrm{can}} := \lambda^2 g_{\mathrm{FS}} + g_M$$

(the sum is defined by the horizontal distribution of the twistor fibration $\mathcal{Z} \to M$ coming from the Levi-Civita connection).

Theorem 5.1 (continued) (Salamon 1982). (M, g) *: positive quaternion Kähler* $\Rightarrow \exists$ *a scaling of the fiber metric s.t. the canonical deformation metric on the twistor space is positive Kähler-Einstein. In fact* g_1^{can} *is Kähler-Einstein.*

- We now compare two families $\{g_\lambda^{\mathrm{CY}}\}$ and $\{g_{\lambda'}^{\mathrm{can}}\}$. The Chow-Yang metric $g_\lambda^{\mathrm{CY}} = \lambda^2(\alpha_1^2 + \alpha_3^2) + {}^tX^0X^0 + {}^tX^1X^1 + {}^tX^2X^2 + {}^tX^3X^3$ and the canonical deformation metric $g_{\lambda'}^{\mathrm{can}} = (\lambda')^2 g_{\mathrm{FS}} + g_M$ coincide if and only if $\lambda = \lambda' = 1$. In particular,

[6] Here we define the metric on $\mathcal{P}$ just by the sum of the base metric on M and the Killing metric on the fiber.

the Chow-Yang metric for $\lambda \neq 1$ do not belong to the family of canonical deformation metrics. The reason is the following. It is well-known that the canonical deformation metric is Kähler-Einstein for a unique suitable partial scaling. Our normalization is that g_1^{can} is Kähler-Einstein (Theorem 5.1). In this case the parallel translation along curves in the $\mathbb{P}^1$-fiber of the twistor fibration $\mathcal{Z} \to M$ preserves the orthogonal complex structure of the twistor space $\mathcal{Z}$ and therefore the "rotation" along the $\mathbb{P}^1$-fiber must belong to $\mathrm{U}(2n)$. In [C-Y], Chow and Yang proved that g_1^{CY} is Kähler-Einstein (see discussions in §6). This means that a canonical deformation metric g_λ^{can} and a Chow-Yang metric $g_{\lambda'}^{\mathrm{CY}}$ coincide if $\lambda = \lambda' = 1$. Suppose next that $\lambda, \lambda' \neq 1$. For g_λ^{can}, there exists an oriented orthonormal frame field for the horizontal subspaces defined globally along a $\mathbb{P}^1$-fiber of the twistor fibration (namely, the constant horizontal frame along the $\mathbb{P}^1$-fiber). We show that such a global object does not exist for the Chow-Yang metric $g_{\lambda'}^{\mathrm{CY}}$. To see this, we fix a $\mathbb{P}^1$-fiber of the twistor fibration. We note that for any value of $\lambda' > 0$, any $\mathbb{P}^1$-fiber is totally geodesic w.r.to the metric $g_{\lambda'}^{\mathrm{CY}}$. Therefore the parallel translation along any curve in the $\mathbb{P}^1$-fiber preserves tangent spaces of the $\mathbb{P}^1$-fiber and therefore preserves the horizontal subspaces. On the other hand, as was shown in [C-Y] by Chow and Yang, the Chow-Yang metric $g_{\lambda'}^{\mathrm{CY}}$ for $\lambda' \neq 1$ is never Kähler (see §6). Therefore the holonomy restricted to the horizontal subspace along any closed curve in the $\mathbb{P}^1$-fiber is not contained in $\mathrm{U}(2n)$. Therefore the set of all parallel translations of a given oriented horizontal orthonormal frame along curves in the $\mathbb{P}^1$-fiber must be identical to $\mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \cong \mathbb{P}^1$. This implies that there exists no smooth oriented horizontal orthonormal frame field defined globally along the $\mathbb{P}^1$-fiber. Indeed, the existence of such a global object would correspond to a global section of the principal $\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n)$-bundle $\mathrm{Sp}(n)\mathrm{Sp}(1) \to \mathrm{Sp}(n)\mathrm{Sp}(1)/\mathrm{Sp}(n)\mathrm{Sp}(1) \cap \mathrm{U}(2n) \cong \mathbb{P}^1$ which never exists. We have thus proved that the Chow-Yang metric $g_{\lambda'}^{\mathrm{CY}}$ is never a canonical deformation metric, because the image under all parallel translations along curves in the $\mathbb{P}^1$-fiber of a given oriented orthonormal frame in the horizontal subspace is a Riemannian invariant and this set has different topological structures for two metrics g_λ^{can} and $g_{\lambda'}^{\mathrm{CY}}$ (here, $\lambda, \lambda' \neq 1$). Indeed, the set has a global horizontal section along the $\mathbb{P}^1$-fiber for the canonical deformation metric g_λ^{can}, while this set does not admit such a global object for the Chow-Yang metric $g_{\lambda'}^{\mathrm{CY}}$.

6. Moving frames on twistor spaces.

• (M^{4n}, g) : a positive quaternion Kähler manifold ($n \geq 2$). $(\mathcal{Z}, J, h)$: $\mathcal{Z}$ is the twistor space of (M, g), J is the orthogonal alm. complex structure and h is the canonical metric with the property that the fiber metric is the Fubini-Study metric with curvature λ^{-2} and the base metric is normalized so that the scalar curvature is the same as $\mathbb{HP}^n$ whose sectional curvatures range in $[1, 4]$. The right multiplication by j defines the canonical identification of $T_m M$ with $\mathbb{C}^{2n}$ given by

$$(x^a + ix^{n+a} + jx^{2n+a} + kx^{3n+a})_{a=1}^n \mapsto (x^a + jx^{2n+a}, x^{n+a} + jx^{3n+a})_{a=1}^n .$$

Pick a point $z \in \mathcal{Z}$ over $m \in M$ which induces this identification.

• Choose the above canonical metric h on $\mathcal{Z}$. ${}^t(\alpha_1, \alpha_3)$: orthonormal coframe in the column real vector notation representing infinitesimal deformation of the unit imaginary quaternion at j. Introduce the column real vectors $X^0 := (x^a)$, $X^1 := (x^{n+a})$, $X^2 := (x^{2n+a})$, $X^3 := (x^{3n+a})$ $(a = 1, \ldots, n)$. Then we have the **real notation** : $\{{}^t(\lambda\alpha_1, \lambda\alpha_3), X^0, X^1, X^2, X^3\}$ which gives an orthonormal basis of $T^*_z\mathcal{Z}$ w.r.to the metric h and the **complex notation w.r.to** J, i.e., a basis of all $(1, 0)$ forms w.r.to the orth. alm. cplx. str. of $\mathcal{Z}$ at z which is given by $\lambda\zeta^0 := \lambda(\alpha_1 + i\alpha_3)$, $Z^1 = X^0 + iX^2$ $(= (x^a + ix^{2n+a}))$ and $Z^2 = X^1 + iX^3$ $(= (x^{n+a} + ix^{3n+a}))$.

• The family of canonical metrics on $\mathcal{Z}$ is expressed as (at $z \in \mathcal{Z}$)

$$\begin{aligned} h &= \lambda^2 \, (\alpha_1^2 + \alpha_3^2) + {}^tX^0 \cdot X^0 + \cdots + {}^tX^3 \cdot X^3 \\ &= \lambda^2 |\zeta^0|^2 + |Z^1|^2 + |Z^2|^2 \ . \end{aligned}$$

• complex notation w.r.to $J \Rightarrow$ integrability of J, Kähler-Einstein property of the canonical metric.

• real notation $\Rightarrow$ curvature computation for non Kähler canonical metrics.

<u>Fundamentals for Moving Frame Differential Calculus on $\mathcal{Z}$</u> :

• 1-st and 2-nd structure equations of (M, g) :

$$\begin{aligned} dX + \Gamma \wedge X &= 0 \\ d\Gamma + \Gamma \wedge \Gamma &= \Omega \end{aligned}$$

• Decomposition of the curvature operator of quaternion Kähler manifolds :

Theorem 6.1 (Alexeevskii 1968, Salamon 1982). *The curvature operator of a quaternion Kähler manifold (M^{4n}, g) $(n \geq 2)$ decomposes as*

$$\Omega \ = \ (S/\widetilde{S}) \, \widetilde{\Omega} \ + \ \Omega'$$

where $\widetilde{\Omega}$ is the curvature operator of $\mathbb{HP}^n$ with the scalar curvature $\widetilde{S}$ and

$$\Omega' \in \mathrm{Sym}^2(\mathrm{sp}(n)) \subset \mathrm{Sym}^2(\Lambda^2 T^* M) \ .$$

Of course Theorem 6.1 is a quaternion Kähler version of the fact that the curvature operator of a self-dual Einstein 4-manifold decomposes into the direct sum of the self-dual part of the Weyl curvature tensor and the $(S/\widetilde{S})$-times the curvature operator of the standard 4-sphere.

• Curvature of the quaternion projective space

$$\mathbb{P}^n(\mathbb{H}) = \frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)} = \frac{\mathrm{Sp}(n+1)/\mathbb{Z}_2}{\mathrm{Sp}(n)\mathrm{Sp}(1)} \ .$$

The $\mathrm{Sp}(1)$ in the middle is a part of the isotropy group at $[1 : 0 : \cdots : 0] \in \mathbb{P}^n(\mathbb{H})$ while $\mathrm{Sp}(1)$ in the right is the image in $\mathrm{SO}(4n)$ of the right action of unit quaternions on $\mathbb{H}^n$ (we must take this difference of the meaning of $\mathrm{Sp}(1)$ into account in the moving frame computation on $\mathbb{P}^n(\mathbb{H})$). Because $\mathbb{P}^n(\mathbb{H})$ is a homogeneous space we can compute the curvature of $\mathbb{P}^n(\mathbb{H})$ from the Maurer-Cartan equation of the big group $\mathrm{Sp}(n+1) \Rightarrow$

$$\begin{aligned}
&d\widetilde{a}_\mu - 2\widetilde{a}_\eta \wedge \widetilde{a}_\nu = 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) \ , \\
&\widetilde{\Omega}^\mu_0 = X^\mu \wedge {}^tX^0 - X^0 \wedge {}^tX^\mu + X^\nu \wedge {}^tX^\eta - X^\eta \wedge {}^tX^\nu \\
&\quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) \ , \\
&\widetilde{\Omega}^\eta_\nu = -X^\mu \wedge {}^tX^0 + X^0 \wedge {}^tX^\mu - X^\nu \wedge {}^tX^\eta + X^\eta \wedge {}^tX^\nu \\
&\quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) \ ,
\end{aligned}$$

(η, μ, ν) being any cyclic permutation of $(1, 2, 3)$. In particular the sectional curvatures of $\mathbb{HP}^n$ range in the interval $[1, 4]$.

Computation on $\mathcal{Z}$:

• Aim : We compute the 1-st structure equations for $d\,{}^t(\zeta^0, {}^tZ^1, {}^tZ^2)$ in the complex notation and $d\,{}^t(\lambda\alpha_1, \lambda\alpha_2, {}^tX^0, {}^tX^1, {}^tX^2, {}^tX^3)$ in the real notation. Then we the corresponding 2-nd structure equations (curvature).

• We remark that even if the description of the Chow-Yang metric is local at 1 point, we can apply the local moving frame computation to the system $\{\alpha_1, \alpha_3, X^0, X^1, X^2, X^3\}$ to compute its curvature. The reason is the following. We continue to work at a point on $z \in \mathcal{Z}$ corresponding to the orthogonal complex structure defined by the right multiplication of j. At points close to z the infinitesimal deformation of the orthogonal complex structures can be expressed as a pair of 1-forms on the $\mathbb{P}^1$-fiber of the twistor fibration, which can be written as $\alpha_1 + O(2)\alpha_2$ and $\alpha_3 + O(2)\alpha_2$, where $O(2)$ represents quantities which are of order 2 w.r.to the distance from the reference point z along the $\mathbb{P}^1$-fiber of the twistor fibration. Furthermore, the α_2 itself is of $O(2)$ around the point z (because it corresponds to the complex structure represented by $j = (0 : 1 : 0) \in \mathbb{P}^1 = \mathrm{Sp}(1)/\mathrm{SO}(2)$). Therefore, the formal computation of the 2-nd structure equation applied to the system $\{\alpha_1, \alpha_3, X^0, X^1, X^2, X^3\}$ gives the curvature.

• The 1-st structure equation on $(M, g) \Leftrightarrow$

$$\begin{aligned}
&dZ^1 + \overline{Z}^2 \wedge \zeta^0 + (A_0 + i(A_2 + \alpha_2)) \wedge Z^1 \\
&\quad + (-A_1 + iA_3) \wedge Z^2 = 0 \ , \\
&dZ^2 - \overline{Z}^1 \wedge \zeta^0 + (A_1 + iA_3) \wedge Z^1 \\
&\quad + (A_0 - i(A_2 - \alpha_2)) \wedge Z^2 = 0 \ .
\end{aligned}$$

• The 2-nd structure equation on (M, g) plus expression (3) $\Rightarrow$

$$\begin{aligned}
&\Omega^\mu_0 = dA_\mu + A_\mu \wedge A_0 + A_\eta \wedge A_\nu + A_0 \wedge A_\mu - A_\nu \wedge A_\eta \\
&\quad + da_\mu - 2a_\eta \wedge a_\nu \ , \\
&\Omega^\eta_\nu = -dA_\mu - A_\eta \wedge A_\nu - A_\mu \wedge A_0 + A_\nu \wedge A_\eta - A_0 \wedge A_\mu \\
&\quad + da_\mu - 2a_\eta \wedge a_\nu \ .
\end{aligned} \tag{4}$$

(μ, η, ν) being any cyclic permutation of $(1,2,3)$.

• Computation of $d\zeta^0$. Formula (4) $\Rightarrow$

$$\Omega_0^\mu + \Omega_\nu^\eta = 2da_\mu - 4a_\eta \wedge a_\nu \ .$$

Combining this with Alexeevskii's decomposition formula we get

$$\begin{aligned} da_\mu - 2a_\eta \wedge a_\nu &= \frac{1}{2}(\Omega_0^\mu + \Omega_\nu^\eta) \\ &= \frac{1}{2}(S/\widetilde{S})(\widetilde{\Omega}_0^\mu + \widetilde{\Omega}_\nu^\eta) + \frac{1}{2}(\Omega'^\mu_0 + \Omega'^\eta_\nu) \\ &= (S/\widetilde{S})(d\widetilde{a}_\mu - 2\widetilde{a}_\eta \wedge \widetilde{a}_\nu) \\ &\text{[because } \Omega' \text{ part does not involve the } a\text{-part]} \\ &= 2(S/\widetilde{S})({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) \end{aligned}$$

and therefore we get

$$\begin{aligned} d\zeta^0 &= d(\alpha_1 + i\alpha_3) \\ &= 2\alpha_2 \wedge \alpha_3 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3) + 2i\alpha_1 \wedge \alpha_2 \\ &\quad + i(d\alpha_3 - 2\alpha_1 \wedge \alpha_2) \\ &= -2i\alpha_2 \wedge \zeta^0 + (S/\widetilde{S})({}^tZ^2 \wedge Z^1 - {}^tZ^1 \wedge Z^2) \ . \end{aligned}$$

We are now ready to right down the 1-st structure equation of the twistor space $(\mathcal{Z}, h)$.

• The 1-st structure equation of $(\mathcal{Z}, h)$ in complex notation:

$$d\begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} = -\begin{pmatrix} 2i\alpha_2 & -(S/\widetilde{S})^t Z^2 & (S/\widetilde{S})^t Z^1 \\ \overline{Z}^2 & A_0 + iA_2 + i\alpha_2 & -A_1 + iA_3 \\ -\overline{Z}^1 & A_1 + iA_3 & A_0 - iA_2 + i\alpha_2 \end{pmatrix} \wedge \begin{pmatrix} \zeta^0 \\ Z^1 \\ Z^2 \end{pmatrix} \ .$$

RHS contains <u>no $(0,2)$-forms</u>
$\Rightarrow$ the alm. cplx. str. defined by the basis $\{\zeta^0, Z^1, Z^2\}$ on $\mathcal{Z}$ of $(1,0)$-forms (i.e., the orthogonal alm. cplx. str. on $\mathcal{Z}$) is <u>integrable</u>.

The matrix in RHS is <u>skew-Hermitian</u>
$\Leftrightarrow$ scaling is chosen s.t. $S/\widetilde{S} = 1$
$\Leftrightarrow$ the canonical metric is <u>Kähler</u> if and only if the scaling of (M, g) is chosen so that $S/\widetilde{S} = 1$ (the fiber Fubini-Study metric is normalized so that the curvature 1).

• 2-nd structure equation $\Rightarrow$ the curvature form Ω of the Kähler metric

$$\zeta^0 \wedge \overline{\zeta}^0 + {}^tZ^1 \wedge \overline{Z}^1 + {}^tZ^2 \wedge \overline{Z}^2$$

is

$$\begin{pmatrix} \begin{matrix} 2\zeta^0 \wedge \overline{\zeta}^0 + {}^tZ^1 \wedge \overline{Z}^1 \\ + {}^tZ^2 \wedge \overline{Z}^2 \end{matrix} & \zeta^0 \wedge {}^t\overline{Z}^1 & \zeta^0 \wedge {}^t\overline{Z}^2 \\ & & \\ Z^1 \wedge \overline{\zeta}^0 & \begin{matrix} \Omega_0^0 + i\,\Omega_0^2 \\ -\overline{Z}^2 \wedge {}^tZ^2 + \zeta^0 \wedge \overline{\zeta}^0 \end{matrix} & \begin{matrix} -\frac{1}{2}\{\Omega_0^1 + \Omega_2^3 - i\,(\Omega_0^3 + \Omega_1^3)\} \\ + {}^t\overline{Z}^2 \wedge {}^tZ^1 \end{matrix} \\ & & \\ Z^2 \wedge \overline{\zeta}^0 & \begin{matrix} \frac{1}{2}\{\Omega_0^1 + \Omega_2^3 + i\,(\Omega_0^3 + \Omega_1^2)\} \\ + \overline{Z}^1 \wedge {}^tZ^2 \end{matrix} & \begin{matrix} \Omega_0^0 + i\,\Omega_1^3 \\ -\overline{Z}^1 \wedge {}^tZ^1 + \zeta^0 \wedge \overline{\zeta}^0 \end{matrix} \end{pmatrix}$$

which is certainly skew-Hermitian. Its Ricci form is

$$\mathrm{Ric}(\Omega) = \mathrm{tr}(\Omega) = 2(n+1)\left\{\zeta^0 \wedge \overline{\zeta}^0 + {}^tZ^1 \wedge \overline{Z}^1 + {}^tZ^2 \wedge \overline{Z}^2\right\},$$

meaning that the Kähler metric

$$\zeta^0 \wedge \overline{\zeta}^0 + {}^tZ^1 \wedge \overline{Z}^1 + {}^tZ^2 \wedge \overline{Z}^2$$

is Kähler-Einstein.

Structure Equations w.r.to Non-Kähler Canonical Metrics

We work on a canonical metric with a scaling parameter $\lambda > 0$:

$$g_\lambda := \lambda^2\,(\alpha_1^2 + \alpha_3^2) + {}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3$$

where $\lambda = 1$ corresponds to the Kähler-Einstein metric.

- 1-st structure equation in real notation is :

$$d\begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & -2\alpha_2 & -\lambda^tX^1 & \lambda^tX^0 & \lambda^tX^3 & -\lambda^tX^2 \\ 2\alpha_2 & 0 & -\lambda^tX^3 & \lambda^tX^2 & -\lambda^tX^1 & \lambda^tX^0 \\ \lambda^{-1}X^1 & \lambda^{-1}X^3 & A_0 & -A_1 & -A_2-\alpha_2 & -A_3 \\ -\lambda^{-1}X^0 & -\lambda^{-1}X^2 & A_1 & A_0 & -A_3 & A_2-\alpha_2 \\ -\lambda^{-1}X^3 & \lambda^{-1}X^1 & A_2+\alpha_2 & A_3 & A_0 & -A_1 \\ \lambda^{-1}X^2 & -\lambda^{-1}X^0 & A_3 & -A_2+\alpha_2 & A_1 & A_0 \end{pmatrix}\begin{pmatrix} \lambda\alpha_1 \\ \lambda\alpha_3 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}.$$

- 2-nd structure equation $d\Gamma_\lambda + \Gamma_\lambda \wedge \Gamma_\lambda = \Omega_\lambda$ gives the curvature

$$\Omega_\lambda = \begin{pmatrix} {\Omega_\lambda}_{-2}^{-2} & {\Omega_\lambda}_{-1}^{-2} & {\Omega_\lambda}_0^{-2} & {\Omega_\lambda}_1^{-2} & {\Omega_\lambda}_2^{-2} & {\Omega_\lambda}_3^{-2} \\ {\Omega_\lambda}_{-2}^{-1} & {\Omega_\lambda}_{-1}^{-1} & {\Omega_\lambda}_0^{-1} & {\Omega_\lambda}_1^{-1} & {\Omega_\lambda}_2^{-1} & {\Omega_\lambda}_3^{-1} \\ {\Omega_\lambda}_{-2}^{0} & {\Omega_\lambda}_{-1}^{0} & {\Omega_\lambda}_0^{0} & {\Omega_\lambda}_1^{0} & {\Omega_\lambda}_2^{0} & {\Omega_\lambda}_3^{0} \\ {\Omega_\lambda}_{-2}^{1} & {\Omega_\lambda}_{-1}^{1} & {\Omega_\lambda}_0^{1} & {\Omega_\lambda}_1^{1} & {\Omega_\lambda}_2^{1} & {\Omega_\lambda}_3^{1} \\ {\Omega_\lambda}_{-2}^{2} & {\Omega_\lambda}_{-1}^{2} & {\Omega_\lambda}_0^{2} & {\Omega_\lambda}_1^{2} & {\Omega_\lambda}_2^{2} & {\Omega_\lambda}_3^{2} \\ {\Omega_\lambda}_{-2}^{3} & {\Omega_\lambda}_{-1}^{3} & {\Omega_\lambda}_0^{3} & {\Omega_\lambda}_1^{3} & {\Omega_\lambda}_2^{3} & {\Omega_\lambda}_3^{3} \end{pmatrix}$$

where

$$\begin{aligned}
&\Omega_{\lambda}{}^{-2}_{-2} = 0\ , \quad \Omega_{\lambda}{}^{-1}_{-1} = 0\ ,\\
&\Omega_{\lambda}{}^{-1}_{-2} = 4\alpha_3 \wedge \alpha_1 + 4({}^tX^2 \wedge X^0 + {}^tX^3 \wedge X^1)\\
&\quad - {}^tX^3 \wedge X^1 - {}^tX^2 \wedge X^0 + {}^tX^1 \wedge X^3 + {}^tX^0 \wedge X^2
\end{aligned}$$

and

$$\begin{aligned}
&\Omega_{\lambda}{}^{0}_{-2} = \lambda^{-1}(X^0 \wedge \alpha_1 + X^2 \wedge \alpha_3), \quad \Omega_{\lambda}{}^{0}_{-1} = \lambda^{-1}(X^0 \wedge \alpha_3 - X^2 \wedge \alpha_1)\\
&\Omega_{\lambda}{}^{1}_{-2} = \lambda^{-1}(X^1 \wedge \alpha_1 + X^3 \wedge \alpha_3), \quad \Omega_{\lambda}{}^{1}_{-1} = \lambda^{-1}(X^1 \wedge \alpha_3 - X^3 \wedge \alpha_1)\\
&\Omega_{\lambda}{}^{2}_{-2} = \lambda^{-1}(-X^0 \wedge \alpha_3 + X^2 \wedge \alpha_1), \quad \Omega_{\lambda}{}^{2}_{-1} = \lambda^{-1}(X^0 \wedge \alpha_1 + X^2 \wedge \alpha_3)\\
&\Omega_{\lambda}{}^{3}_{-2} = \lambda^{-1}(-X^1 \wedge \alpha_3 + X^3 \wedge \alpha_1), \quad \Omega_{\lambda}{}^{3}_{-1} = \lambda^{-1}(X^1 \wedge \alpha_1 + X^3 \wedge \alpha_3)\\
&\Omega_{\lambda_0}{}^{-2} = \lambda(\alpha_1 \wedge {}^tX^0 + \alpha_3 \wedge {}^tX^2), \quad \Omega_{\lambda_0}{}^{-1} = \lambda(\alpha_3 \wedge {}^tX^0 - \alpha_1 \wedge {}^tX^2)\\
&\Omega_{\lambda_1}{}^{-2} = \lambda(\alpha_1 \wedge {}^tX^1 + \alpha_3 \wedge {}^tX^3), \quad \Omega_{\lambda_1}{}^{-1} = \lambda(\alpha_3 \wedge {}^tX^1 - \alpha_1 \wedge {}^tX^3)\\
&\Omega_{\lambda_2}{}^{-2} = \lambda(-\alpha_3 \wedge {}^tX^0 + \alpha_1 \wedge {}^tX^2), \quad \Omega_{\lambda_2}{}^{-1} = \lambda(\alpha_1 \wedge {}^tX^0 + \alpha_3 \wedge {}^tX^2)\\
&\Omega_{\lambda_3}{}^{-2} = \lambda(-\alpha_3 \wedge {}^tX^1 + \alpha_1 \wedge {}^tX^3), \quad \Omega_{\lambda_3}{}^{-1} = \lambda(\alpha_1 \wedge {}^tX^1 + \alpha_3 \wedge {}^tX^3)\ .
\end{aligned}$$

and

$$\begin{aligned}
&\Omega_{\lambda}{}^0_0 = \Omega^0_0 - X^1 \wedge {}^tX^1 - X^3 \wedge {}^tX^3, \quad \Omega_{\lambda}{}^1_1 = \Omega^0_0 - X^0 \wedge {}^tX^0 - X^2 \wedge {}^tX^2\\
&\Omega_{\lambda}{}^2_2 = \Omega^0_0 - X^1 \wedge {}^tX^1 - X^3 \wedge {}^tX^3, \quad \Omega_{\lambda}{}^3_3 = \Omega^0_0 - X^0 \wedge {}^tX^0 - X^2 \wedge {}^tX^2
\end{aligned}$$

and

$$\begin{aligned}
&\Omega_{\lambda}{}^1_0 = \Omega^1_0 + X^0 \wedge {}^tX^1 + X^2 \wedge {}^tX^3 - (d\alpha_1 - 2\alpha 2 \wedge \alpha_3)\\
&\Omega_{\lambda}{}^2_0 = \Omega^2_0 + X^3 \wedge {}^tX^1 - X^1 \wedge {}^tX^3 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2\\
&\Omega_{\lambda}{}^3_0 = \Omega^3_0 - X^2 \wedge {}^tX^1 + X^0 \wedge {}^tX^3 - (d\alpha_3 - 2\alpha_1 \wedge \alpha_2)\\
&\Omega_{\lambda}{}^2_1 = \Omega^2_1 - X^3 \wedge {}^tX^0 + X^1 \wedge {}^tX^2 + (d\alpha_3 - 2\alpha_1 \wedge \alpha_2)\\
&\Omega_{\lambda}{}^3_1 = \Omega^3_1 + X^2 \wedge {}^tX^0 - X^0 \wedge {}^tX^2 - (d\alpha_2 - 2\alpha_3 \wedge \alpha_1) + d\alpha_2\\
&\Omega_{\lambda}{}^3_2 = \Omega^3_2 + X^2 \wedge {}^tX^3 + X^0 \wedge {}^tX^1 + (d\alpha_1 - 2\alpha_2 \wedge \alpha_3)\ .
\end{aligned}$$

Alexeevskii's decomposition formula $\Rightarrow$

$$\begin{aligned}
\Omega^\mu_0 &= \widetilde{\Omega}^\mu_0 + \Omega'^\mu_0\\
&= X^\mu \wedge {}^tX^0 - X^0 \wedge {}^tX^\mu + X^\nu \wedge {}^tX^\eta - X^\eta \wedge {}^tX^\nu\\
&\quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) + \Omega'^\mu_0\ ,\\
\Omega^\eta_\nu &= \widetilde{\Omega}^\eta_\nu + \Omega'^\eta_\nu\\
&= -X^\mu \wedge {}^tX^0 + X^0 \wedge {}^tX^\mu - X^\nu \wedge {}^tX^\eta + X^\eta \wedge {}^tX^\nu\\
&\quad + 2({}^tX^\mu \wedge X^0 + {}^tX^\eta \wedge X^\nu) + \Omega'^\eta_\nu\ ,
\end{aligned}$$

Ricci tensor of a non-Kähler Chow-Yang metric.

Proposition 6.1. *The "hyper-Kähler part" ${\Omega'}^{\mu}_{\nu}$ has no contribution to the Ricci tensor. Therefore we can ignore the ${\Omega'}^{\mu}_{\nu}$-part in the computation of the Ricci tensor.*

The dependency on the point of the full curvature does not contribute to the Ricci map $g \mapsto \mathrm{Ric}(g)$.

• Set up : $\{\xi_{-2}, \xi_{-1}, \xi_0, \dots, \xi_3\}$: the frame on $\mathcal{Z}$ dual to the coframe $\{a_1, a_3, X^0, \dots, X^3\}$ (unitary w.r.to the KE metrix) $\Rightarrow$

$$\{\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}, \xi_0, \dots, \xi_3\}$$

is the frame dual to the coframe

$$\{\lambda a_1, \lambda a_3, X^0, \dots, X^3\} \ .$$

• Computation of the Ricci tensor. Using the formula

$$\mathrm{Ric}(e_i, e_j) = \sum_{k=1}^{\dim} g\,(\,\Omega^j_k\,(e_i, e_k)\,e_j,\ e_j)\ ,$$

we get

$$\begin{aligned}
&\mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-2}) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-1}, \lambda^{-1}\xi_{-1})\\
&= \frac{4}{\lambda^2} + 4n\ ,\\
&\mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \lambda^{-1}\xi_{-1}) = 0\ ,\\
&\quad \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \xi_0) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \xi_1) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \xi_2)\\
&= \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-2}, \xi_3) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-1}, \xi_0) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-1}, \xi_1)\\
&= \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-1}, \xi_2) = \mathrm{Ric}_\lambda(\lambda^{-1}\xi_{-1}, \xi_3)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Ric}(\xi_0, \xi_0) = \mathrm{Ric}_\lambda(\xi_1, \xi_1) = \mathrm{Ric}_\lambda(\xi_2, \xi_2) = \mathrm{Ric}_\lambda(\xi_3, \xi_3)\\
&\quad = \frac{2}{\lambda^2} + (4n+2)\ ,\\
&\mathrm{Ric}(\xi_0, \xi_1) = \mathrm{Ric}_\lambda(\xi_0, \xi_2) = \mathrm{Ric}_\lambda(\xi_0, \xi_3) = \mathrm{Ric}_\lambda(\xi_1 . \xi_2)\\
&\ = \mathrm{Ric}_\lambda(\xi_1, \xi_3) = \mathrm{Ric}_\lambda(\xi_2, \xi_3)\\
&\ = 0\ .
\end{aligned}$$

Theorem 6.2. *The Ricci tensor of the metric g_λ on the twistor space $\mathcal{Z}$ is given by the formula*

$$\begin{aligned}
\mathrm{Ric}_\lambda \ =\ & 4\,(1 + n\lambda^2)\,(\alpha_1^2 + \alpha_3^2)\\
& + 2\,(\lambda^{-2} + 2n + 1)\,({}^tX^0 \cdot X^0 + {}^tX^1 \cdot X^1 + {}^tX^2 \cdot X^2 + {}^tX^3 \cdot X^3)\ .
\end{aligned}$$

In other words,

$$\mathrm{Ric}_\lambda = 2\lambda^{-2}\left\{1+(2n+1)\lambda^2\right\} g_{\sqrt{\frac{2\lambda^2(1+n\lambda^2)}{1+(2n+1)\lambda^2}}} \, .$$

This means that the 2-parameter family

$$\mathcal{F} = \{\rho\, g_\lambda\}_{\rho,\lambda>0}$$

of the (scaled) 2-parameter family of the canonical metrics on $\mathcal{Z}$ is stable under the Ricci map.

Remark 6.3. If $\lambda = 1$ we get

$$\mathrm{Ric}_1 \;=\; 4\,(n+1)\, g_1 \; .$$

Remark 6.4 (orbifold case). Of course we can construct locally irreducible positive quaternion Kähler orbifolds which are uniformized by one of the Wolf spaces. On the other hand, many examples of non locally symmetric positive quaternion Kähler orbifolds are constructed in [G-L]. Here we remark that the moving frame computation in §2 does not necessarily generalize to positive quaternion Kähler orbifold case. Indeed, given a locally irreducible positive quaternion Kähler orbifold, the attempt constructing its twistor space with its complex structure may not work. Moreover, even if the orbifold version of the twistor space exists, the orbifold version of the Chow-Yang metric is not defined.

Here we explain the reason.

If we take a local uniformization of the orbifold along the locus of orbifold singularities, we locally get a non-singular irreducible quaternion Kähler manifold with a finite group G acting isometrically preserving the local quaternion Kähler structure. In the case where G operates on the local holonomy reduction $\mathcal{P}_{\mathrm{loc}}$ of the oriented orthonormal frame bundle, we can construct the orbifold version of the twistor space to this case just by working equivariantly. However, in the case where the group G does not operate on $\mathcal{P}_{\mathrm{loc}}$, we cannot generalize the arguments in §2. Indeed, the action of G can be defined in the $\mathrm{SO}(4n)$-principal bundle of the full space of oriented orthonormal frames of M and the space on which G can act consists of two copies of $\mathcal{P}_{\mathrm{loc}}$ whose fibers over the loci on which the G-action is not free coincide. In other wards, the holonomy of the orbifold under question along a loop which approaches to the orbifold singular loci and go around it and come back is not contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$ and the holonomy group becomes a disconnected subgroup of $\mathrm{SO}(4n)$ whose identity component is a subgroup of $\mathrm{Sp}(n)\mathrm{Sp}(1)$. Therefore, in this case, the twistor space cannot be defined as a usual Kähler orbifold.

Suppose next that G operates on the local holonomy reduction $\mathcal{P}_{\mathrm{loc}}$ of the oriented orthonormal frame bundle. In this case we can construct the orbifold version of the twistor space. To see what happens to the construction of the orbifold Chow-Yang metric, we work on the local uniformization level. The Chow-Yang metric at a point z in the $\mathbb{P}^1$-fiber over $m \in M$ of the twistor fibration $Z \to M$ in the local uniformization level, the coframe, say, $\{\alpha_1, \alpha_3, X^0, X^1, X^2, X^3\}$ is chosen so that $\{X^0, X^1, X^2, X^3\}$ is a unitary basis of T_mM with respect to the orthogonal complex structure corresponding to z. However, in the local uniformization level, we must work equivariantly with respect to the finite group action. This group action identifies different orthogonal complex structures and induces in general a non trivial rotation in the space $\mathbb{P}^1_m$ of orthogonal complex structures of T_mM in the local uniformization level. Therefore we cannot define the Chow-Yang metric in the equivariant way and this implies that the orbifold Chow-Yang metric is not defined in general (the case where the Chow-Yang metric is defined equivariantly corresponds to orbifolds uniformlized by the Wolf spaces).

We thus conclude that the arguments in §2 cannot be generalized to quaternion Kähler orbifolds.

For comparison, we compute (using O'Neill's formula) the Ricci tensor of the canonical deformation metric g^{can}_λ. The result is

$$\mathrm{Ric}(g^{\mathrm{can}}_\lambda) = (1+n\lambda^4)g_{FS} + (n+2-\lambda^2)g_M \ .$$

Therefore g^{can}_λ is Einstein iff $\lambda^2 = 1$ and $\lambda^2 = \frac{1}{n+1}$. The case $\lambda^2 = 1$ corresponds to the submersion metric coming from $\mathcal{P} \to \mathcal{Z}$ which is Kähler-Einstein. Another Einstein metric (corresponding to the case $\lambda^2 = \frac{1}{n+1}$) is non-Kähler. In this case the Ricci flow equation $\partial_t g = -2\mathrm{Ric}_g$ reduces to the following system of ODE's on the family of canonical deformation metrics $\{\rho g^{\mathrm{can}}_\lambda\}_{\lambda,\rho}$:

$$\begin{cases} \dfrac{d\lambda^2}{dt} = -\dfrac{2}{\rho}(\lambda^2-1)\{(n+1)\lambda^2-1\} \\ \dfrac{d\rho}{dt} = -2(n+2-\lambda^2) \ . \end{cases}$$

It turns out that the behavior is completely different from the Ricci flow defined on the family of Chow-Yang metrics $\{\rho g^{\mathrm{CY}}_\lambda\}_{\lambda,\rho}$ (see Theorem 7.1).

7. Ricci flow.

The family $\mathcal{F}$ is stable under the scaling by positive numbers and the convex sum. Therefore the family $\mathcal{F}$ is stable under the Ricci flow equation. Here, for a time dependent metrics $g(x,t)$ is said to satisfy the Ricci flow equation if the evolution equation

$$\partial_t \, g \; = \; -2 \, \mathrm{Ric}_g$$

holds.

Theorem 7.1. *(1) The Ricci flow equation*

$$\partial_t g = -2 \, \mathrm{Ric}_g$$

on the twistor space $\mathcal{Z}$ with initial metric in the family $\mathcal{F}$ reduces to the system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt} \left(\rho(t) \lambda^2(t) \right) \; = \; -8 \left(1 + n \, \lambda^2(t) \right) \, , \\ \dfrac{d}{dt} \, \rho(t) \; = \; -4 \left(\lambda(t)^{-2} + 2n + 1 \right) \, . \end{cases}$$

(2) For any initial metric at time $t = 0$ in the homothetically extended family of Chow-Yang metrics on $\mathcal{Z}$, the system of ordinary differential equations (12) has a solution defined on $(-\infty, T)$, i.e. the solution is extended for all negative reals (such a solution is called an **ancient solution***) and extinct at some finite time T (i.e., as $t \to T$ the solution shrinks the space and become extinct at time T). The extinction time T depends on the choice of the initial metric.*

(3) Suppose that $\rho(0) = 1$. If $\lambda(0) = 1$, then the metric remains Kähler-Einstein ($\lambda(t) \equiv 1$) and the solution evolves just by homothety $\rho(t) = 1 - 4(n+1)t$ (in this case $T = \frac{1}{4(n+1)}$).

If $\lambda(0) < 1$, then

$$\lim_{t \to -\infty} \lambda(t) = 1 \quad \lim_{t \to -\infty} \rho(t) = \infty \, ,$$
$$\lim_{t \to T} \lambda(t) = 0 \, , \; \lim_{t \to T} \rho(t) = 0 \, .$$

If $\lambda(0) > 1$, then

$$\lim_{t \to -\infty} \lambda(t) = 1 \, , \quad \lim_{t \to -\infty} \rho(t) = \infty \, ,$$
$$\lim_{t \to T} \lambda(t) = \infty \, , \; \lim_{t \to T} \rho(t) = 0 \, , \; \lim_{t \to T} \rho(t) \lambda^2(t) = 0 \, .$$

Suppose that $\lambda(0) \neq 1$. Then, as t becomes larger in the future direction, the deviation $|1 - \lambda(t)|$ of the solution from being Kähler-Einstein becomes larger as well. As t becomes larger in the past direction, then the solution becomes backward asymptotic to the solution in the case of $\lambda(0) = 1$, i.e., the Kähler-Einstein metric is the asymptotic soliton of the Ricci flow under consideration.

(4) Suppose that $\lambda(0) < 1$. Then the Gromov-Hausdorff limit of the Ricci flow solution as $t \to T$, scaled with the factor $\rho(t)^{-1}$, is the original quaternion Kähler metric on M.

(5) Suppose that $\lambda(0) > 1$. Then the Gromov-Haussdorff limit of the Ricci flow solution as $t \to T$, scaled with the factor $\rho(t)^{-1}$, is the sub-Riemmanian metric defined on the horizontal distribution of the twistor space $\mathcal{Z}$ which projects isometrically to the original quaternion Kähler metric on M.

Application of Theorem 6.2 and 7.1.

Theorem 6.1 and 7.1 imply that the 2-parameter family $\mathcal{F}$ is foliated by the trajectories of the Ricci flow solutions all of which are ancient solutions.

We can draw the picture of this foliation.

Applying the curvature derivative estimates for the Ricci flow due to Bando and Shi ([B], [Shi], see also presentation in [C-K]) to the ancient solutions in Theorem 7.1 (together with the curvature computation in §6), we get the following (for a proof of Theorem 7.2, see [K-O1] arXiv:0801.2605 [math.DG].) :

Theorem 7.2. *We have the limit formula*

$$\lim_{\lambda\to\infty} |\nabla^{g_\lambda^{\mathrm{CY}}} \mathrm{Rm}^{g_\lambda^{\mathrm{CY}}}|_{g_\lambda^{\mathrm{CY}}} = 0 \ .$$

This supports the LeBrun-Salamon Conjecture. If LeBrun-Salamon Conjecture is true, we must have

$$\nabla \mathrm{Rm} = 0$$

for the original quaternion Kähler metric g on M^{4n}.

As g_λ tends to the sub-Riemannian metric on $\mathcal{Z}$ which covers the original (M^{4n}, g) $(n \geq 2)$ isometrically, there is a possibility that the limit formula

$$\lim_{\lambda\to\infty} |\nabla^{g_\lambda^{\mathrm{CY}}} \mathrm{Rm}^{g_\lambda^{\mathrm{CY}}}|_{g_\lambda^{\mathrm{CY}}} = 0$$

in Theorem 7.2 implies the LeBrun-Salamon conjecture

$$\nabla \mathrm{Rm} = 0 \ ,$$

i.e., the original (M^{4n}, g) is isometric to one of the Wolf spaces. In fact, we can prove the following Theorem 7.3, full detail of which can be found in [K-O1], arXiv:0801.2605 [math.DG].

Theorem 7.3. *Any irreducible positive quaternion Kähler manifold (M^{4n}, g) is isometric to one of the Wolf spaces.*

Outline of Proof. We choose an orthonormal basis (e_A) of the tangent space (T_mM, g_m) $(m \in M)$ and extend it to an orthonormal frame on a neighborhood of m by parallel transportation along geodesics emanating from m. This defines a $(4n)$-dimensional surface S centered at $(e_A) \in \mathcal{P}$ ($\mathcal{P}$ being the holonomy reduction of the principal bundle of orthonormal frames of M) which is transversal to the vertical foliation. This determines a $(4n)$-dimensional surface S' in the twistor space $\mathcal{Z}$ centered at a point $\widetilde{m}$ on a $\mathbb{P}^1$-fiber over m, which is transversal to the $\mathbb{P}^1$-fibration. The covariant derivative of the curvature tensor at m is computed by differentiating the components of the curvature tensor w.r.to the orthonormal frames represented by points of S (identified with S' in $\mathcal{Z}$) in the direction of a horizontal tangent vector at $\widetilde{m}$ (identified with a tangent vector of M at m). In §6, we computed the curvature form of the metric $g_\lambda = \lambda^2(\alpha_1^2+\alpha_3^2)+\sum_{i=0}^3 {}^tX^i\cdot X^i$ on $\mathcal{Z}$. For our purpose, we need:

$$
\begin{aligned}
\Omega_{\lambda}{}_0^0 &= \Omega_0^0 - X^1\wedge {}^tX^1 - X^3\wedge {}^tX^3\\
\Omega_{\lambda}{}_1^1 &= \Omega_0^0 - X^0\wedge {}^tX^0 - X^2\wedge {}^tX^2\\
\Omega_{\lambda}{}_2^2 &= \Omega_0^0 - X^1\wedge {}^tX^1 - X^3\wedge {}^tX^3\\
\Omega_{\lambda}{}_3^3 &= \Omega_0^0 - X^0\wedge {}^tX^0 - X^2\wedge {}^tX^2
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_{\lambda}{}_0^1 &= \Omega_0^1 + X^0\wedge {}^tX^1 + X^2\wedge {}^tX^3 - (d\alpha_1 - 2\alpha_2\wedge\alpha_3)\\
\Omega_{\lambda}{}_0^2 &= \Omega_0^2 + X^3\wedge {}^tX^1 - X^1\wedge {}^tX^3 - (d\alpha_2 - 2\alpha_3\wedge\alpha_1)\\
&\quad + d\alpha_2\\
\Omega_{\lambda}{}_0^3 &= \Omega_0^3 - X^2\wedge {}^tX^1 + X^0\wedge {}^tX^3 - (d\alpha_3 - 2\alpha_1\wedge\alpha_2)\\
\Omega_{\lambda}{}_1^2 &= \Omega_1^2 - X^3\wedge {}^tX^0 + X^1\wedge {}^tX^2 + (d\alpha_3 - 2\alpha_1\wedge\alpha_2)\\
\Omega_{\lambda}{}_1^3 &= \Omega_1^3 + X^2\wedge {}^tX^0 - X^0\wedge {}^tX^2 - (d\alpha_2 - 2\alpha_3\wedge\alpha_1)\\
&\quad + d\alpha_2\\
\Omega_{\lambda}{}_2^3 &= \Omega_2^3 + X^2\wedge {}^tX^3 + X^0\wedge {}^tX^1 + (d\alpha_1 - 2\alpha_2\wedge\alpha_3)\ .
\end{aligned}
$$

Taking the component in the X^i $(i=0,1,2,3)$ direction of the curvature tensor and taking the covariant derivative in the X^i $(i=0,1,2,3)$ direction, we immediately conclude that the covariant derivatives of the X^i $(i=0,1,2,3)$ part of the curvature tensor of the metric g_λ^{CY} of the twistor space $\mathcal{Z}$ at $\widetilde{m}$ in the horizontal direction is equal to the covariant derivative in the corresponding direction of the curvature tensor of the quaternion Kähler manifold (M,g) under question. On the other hand, we have from Theorem 7.2 the limit formula

$$
\lim_{\lambda\to\infty} |\nabla^{g_\lambda^{\mathrm{CY}}}\mathrm{Rm}^{g_\lambda^{\mathrm{CY}}}|_{g_\lambda^{\mathrm{CY}}} = 0\ .
$$

This implies that the curvature tensor of the positive quaternion Kähler manifold (M,g) must satisfy the condition $\nabla R\equiv 0$ from the beginning. This implies that (M,g) is a symmetric space. Since we assumed that (M,g) is irreducible, (M,g) must be isometric to one of the Wolf spaces. $\square$

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