Extremals for the Sobolev inequality on the quaternionic Heisenberg group and the quaternionic contact Yamabe problem

Stefan Ivanov

based on math.DG/0611658, math.DG/0703044 - joint works with Ivan Minchev & Dimiter Vassilev,
and arXiv:0707.1289 - joint with Dimiter Vassilev,

Hamburg, July 2008

Theorem (G.Folland & E.Stein)

Let $\Omega \subset \mathbf{G}$ *be an open set in a group of Heisenberg type (Carnot group) of homogeneous dimension Q. For any* $1 < p < Q$ *there exists* $S_p = S_p(\mathbf{G}) > 0$ *such that for* $u \in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{p^*} \, dH(g)\right)^{1/p^*} \leq \; S_p \, \left(\int_\Omega |Xu|^p \, dH(g)\right)^{1/p}.$$

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The best constant is achieved.

The Folland-Stein inequality. $p^* = \frac{pQ}{Q-p}$

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- When $p = 2$, -the problem reduces to the solvability of the Yamabe equation

$$\sum_{i=1}^m X_i^2 u = -u^{\frac{Q+2}{Q-2}}.$$

- Zero dimensional center - $\mathbf{G} = \mathbb{R}^n$.
  The problem can be translated to the standard sphere S^n via the stereographic map - leads to involve Riemannian geometry - the Riemannian Yamabe problem.

Let (M, g) - compact, Riemannian manifold, $2^* = \frac{2n}{n-2}$. If $\bar{g} = u^{4/(n-2)}g$, then

$$4\frac{n-1}{n-2}\triangle u - \text{Scal}\cdot u = -\overline{\text{Scal}}\cdot u^{2^*-1}.$$

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- Note that extremals and the best constant can be found directly, not involving Riemannian geometry and the Riemannian Yamabe.

The classical Obata

The main idea is to replace the non-linear Yamabe on S^n with a geometrical system of equations which can be solved-namely conformal deformations preserving the Einstein condition.

Theorem (Aubin,Talenti, Obata)

Let (S^n, g_{st}) be the unit sphere in $\mathbb{R}^{n+1}$. If g is a Riem. metric, $g = \phi^2 g_{st}$, and $Scal_g = S = const$, then up to a homothety g is obtained from g_{st} by a conformal diffeo of the sphere, i.e.,

$$\exists \Phi \in Diff(S^n) \text{ s.t. } Sg = \Phi^* g_{st}$$

Furthermore, $\Phi = exp(tX)$, $X = \nabla f$, $f = a_o x_o + \cdots + a_n x_n|_{S^n}$.

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"Proof" (Lee & Parker) $\bar{g}$ is Einstein. i.e., $0 = \overline{Ric_o} = Ric_o + \frac{n-2}{\phi}(\nabla^2\phi)_o$. Thus, $(\nabla^2\phi)_o = -\frac{\phi}{n-2} Ric_o$. Using $2\nabla^*(Ric_o) = \nabla S = 0$, from the contracted Bianchi and S=const, it follows

$$div\ Ric_o(\nabla\phi, .) = -\frac{\phi}{n-2}|Ric_o|^2.$$

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Theorem (N. Trudinger, Th. Aubin, R. Schoen; A. Bahri)

Let $(M^n, \bar{g})$, $n \geq 3$, be a compact Riemannian manifold. There is a $g \in [\bar{g}]$, s.t., $Scal_g = const$.

One-dimensional center-The CR Obata

One-dimensional center - (complex) Heisenberg group $\mathbf{G} = \mathbb{C}^n \times \mathbb{R}$.

- -It is not known a direct solution to the L^2-Folland-Stein inequality.
- - it is solved by D.Jerison and J.Lee using CR-geometry and the CR-Yamabe problem involving Tanaka-Webster connection and its scalar curvature.

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Theorem (D. Jerison & J. Lee '88)

If θ is the contact form of a pseudo-Hermitian structure proportional to the standard contact form $\bar{\theta}$ on the unit sphere in $\mathbb{C}^{n+1}$ and $Scal_\theta =$const, then up to a multiplicative constant $\theta = \Phi^* \, \bar{\theta}$ with Φ a CR automorphism of the sphere.

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CR automorphism of the sphere are relatively easy to described due to

Theorem (J. Lee '88)

If $(M, \bar\theta)$ is pseudo-Einstein, then $\theta = e^{2u}\bar\theta$ is pseudo-Einstein iff u is CR-pluriharmonic on M.

The CR Yamabe problem II

The CR-Yamabe problem is: Given a compact strongly pseudo-convex CR manifold $(M^{2n+1}, \theta) \subset \mathbb{C}^{n+1}$ find a smooth function f such that $\bar{\theta} = e^f \theta$ has constant pseudohermitian scalar curvature.

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Theorem (D. Jerison & J. Lee '87-'89)

a) $\Upsilon([\theta]) \leq \Upsilon(S^{2n+1})$, *where* $S^{2n+1} \subset \mathbb{C}^{n+1}$ *is the sphere with its standard CR structure. If* $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$, *then the Yamabe equation has a solution. [D. Jerison & J. Lee '87]*

b) *If* $n \geq 2$ *and M is not locally CR equivalent to* S^{2n+1}, *then* $\Upsilon([\theta]) < \Upsilon(S^{2n+1})$. *[D. Jerison & J. Lee '89]*

$$Y(\theta_\epsilon) = \begin{cases} Y(S^{2n+1})\left(1 - c_n|S(q)|^2\epsilon^4\right) + \mathcal{O}(\epsilon^5), & n \geq 2; \\ Y(S^5)\left(1 - c_2|S(q)|^2\epsilon^4 \ln \epsilon\right) + \mathcal{O}(\epsilon^4), & n = 2. \end{cases}$$

c) *If* $n = 1$ *or M is locally CR equivalent to* S^{2n+1}, *then the Yamabe equation has a solution. [R. Yacoub '01, N. Gamara & R. Yacoub, 01]*

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S is the Chern-Moser tensor and one of the key point is the use of the Chern-Moser theorem

Theorem (Chern-Moser '74)

A $(2n+1), n > 1$*-dimensional CR manifold is locally CR-equivalent to the sphere exactly when the Chern-Moser tensor vanishes,* $S = 0$.

Three-dimensional center-Quaternionic Heisenberg Group $G(\mathbb{H})$

- $\mathbb{H}$-quaternions, $q = t + ix + jy + kz$, where $t, x, y, z \in \mathbb{R}$ and $i,\ j,\ k$ satisfy the multiplication rules
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Left-invariant horizontal vector fields

$$T_\alpha = \frac{\partial}{\partial t_\alpha} + 2x^\alpha \frac{\partial}{\partial x} + 2y^\alpha \frac{\partial}{\partial y} + 2z^\alpha \frac{\partial}{\partial z} \quad X_\alpha = \frac{\partial}{\partial x_\alpha} - 2t^\alpha \frac{\partial}{\partial x} - 2z^\alpha \frac{\partial}{\partial y} + 2y^\alpha \frac{\partial}{\partial z}$$
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The 'quaternionic contact Yamabe equation' on $\boldsymbol{G}\,(\mathbb{H})$ is

$$\sum_{\alpha=1}^{n} \left(T_\alpha^2 + X_\alpha^2 + Y_\alpha^2 + Z_\alpha^2\right) u = -\frac{n+1}{4(n+2)} u^{2^*-1}.Const.$$

Theorem (Folland and Stein)

Let $\mathbf{G} = \mathbb{H} \times Im\,\mathbb{H}$ *and* $\Omega \subset \mathbf{G}$. *There is* $S_2 = S_2(\mathbf{G}) > 0$, *such that, for* $u \in C_o^\infty(\Omega)$

$$\left(\int_\Omega |u|^{2^*} \, dH(g) \right)^{1/2^*} \leq S_2 \left(\int_\Omega |\nabla u|^2 \, dH(g) \right)^{1/2}, \qquad 2^* = 5/4.$$

Folland-Stein inequality in Dimension Seven-main result

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Theorem (I/, I. Minchev, D. Vassilev)

Let $\mathbf{G} = \mathbb{H} \times Im\,\mathbb{H}$. *The best constant in the* L^2 *Folland-Stein embedding theorem is*

$$S_2 = \frac{2\sqrt{3}}{\pi^{3/5}}.$$

An extremal is given by the function

$$F(g) = \frac{2^{11}\sqrt{3}}{\pi^{3/5}} \left[(1 + |q|^2)^2 + \|\omega|^2) \right]^{-2}, \quad (q, \omega) \in \boldsymbol{G}\,(\mathbb{H}).$$

Any other non-negative extremal is obtained from F *by translations* $\tau_{(q_0,\omega_0)} F = F(q_0 + q, \omega_0 + \omega)$ *and dilations* $F_\lambda = \lambda^4 F(\lambda q, \lambda^2 \omega), \lambda > 0$.

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i) *a 2-sphere bundle $\mathbb{Q}$ over M of almost complex structures, such that, we have $\mathbb{Q} = \{aI_1 + bI_2 + cI_3 : \ a^2 + b^2 + c^2 = 1\}$, where the almost complex structures $I_s : H \to H, \quad I_s^2 = -1, \quad s = 1, 2, 3,$ satisfy the commutation relations of the imaginary quaternions $I_1 I_2 = -I_2 I_1 = I_3$;*

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ii) *H is the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in $\mathbb{R}^3$ and the following compatibility condition holds*
$$2g(I_s X, Y) = d\eta_s(X, Y), \quad s = 1, 2, 3, \quad X, Y \in H.$$

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Example 3-Sasakian manifolds M: The cone $C = M \times \mathbb{R}$ with the metric $g_{con} = dt^2 + t^2 g$ is a hyperkähler metric.

- Given η (and H) there exists at most one triple of a.c.str. and metric g that are compatible.
- Rotating η we obtain the same qc-structure.
- Conformal transformations $\eta = (\eta_1, \eta_2, \eta_3)$, $\mu \in \mathcal{C}^\infty(M)$, $\mu > 0$, $\Psi \in \mathcal{C}^\infty\,(M : SO(3))$.

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Theorem (O. Biquard)

Under the above conditions and $n > 1$, there exists a unique supplementary distribution V of H in TM and a linear connection ∇ on M, s.t.,

1. *V and H are parallel*
2. *g and Q are parallel*
3. *torsion $T(A,B) = \nabla_A B - \nabla_B A - [A,B]$ satisfies*
   - $\forall X, Y \in H, \quad T_{X,Y} = -[X,Y]|_V \in V$
   - $\forall \xi \in V, \quad T_\xi := (X \mapsto (T_{\xi,X})_H) \in (sp(n) + sp(1))^\perp$

- Note: V is generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$

$$\eta_s(\xi_k) = \delta_{sk}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \qquad (\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$$

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- If the dimension of M is seven, $n = 1$, the above conditions do not always hold. Duchemin shows that if we assume, in addition, the existence of Reeb vector fields as above, then there is a connection as before. Henceforth, by a qc structure in dimension 7 we shall mean a qc structure satisfying the Reeb conditions

- Contact 3-form on the Quaternionic Heizenberg group $\boldsymbol{G}\,(\mathbb{H})$.
  $\tilde{\Theta} = (\tilde{\Theta}_1, \tilde{\Theta}_2, \tilde{\Theta}_3) = \frac{1}{2}\,(d\omega - q\cdot d\bar{q} + dq\cdot\bar{q})$ or

$$\tilde{\Theta}_1 = \frac{1}{2}\,dx - x^{\alpha}dt^{\alpha} + t^{\alpha}dx^{\alpha} - z^{\alpha}dy^{\alpha} + y^{\alpha}dz^{\alpha}$$
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- Left-invariant Reeb (vertical) vector fields ξ_1, ξ_2, ξ_3 are

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- $(\boldsymbol{G}\,(\mathbb{H}), \tilde{\theta})$- the flat model.

- 3-Sasakian sphere: Contact 3-form on the sphere $S^{4n+3} = \{|q|^2 + |p|^2 = 1\} \subset \mathbb{H}^n \times \mathbb{H}$,

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- $\mathcal{C}^*\,\tilde{\Theta} \;=\; \frac{1}{2\,|1+p\,|^2}\,\lambda\,\tilde{\eta}\,\bar{\lambda},\quad$ λ-unit quaternion (eg. of *conformal quaternionic contact map*).

- curvature: $R(A,B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C, \quad R(A,B,C,D) = g(R(A,B)C,D)$;
- qc-Ricci tensor: $Ric(X,Y) = tr_H\{Z \mapsto \mathcal{R}(Z,X)Y\} = R(e_a,X,Y,e_a)$ for $e_a, X, Y \in H$
- qc-Ricci 2-forms $\rho_s(X,Y) = R(X,Y,e_a,I_s e_a)$
- qc-scalar curvature: $\mathrm{Scal} = tr_H\, Ric = Ric(e_a,e_a)$.
- qc-Einstein condition:

$$Ric(X,Y) = \frac{Scal}{4n} g(X,Y), \quad X,Y \in H$$

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Lemma (O. Biquard '99)

If $\bar\eta = u^{4/(Q-2)} \eta$, *then*

$$4 \frac{Q+2}{Q-2} \triangle u - u\, Scal = -u^{2^*-1}\, \overline{Scal},$$

where $\triangle u = tr_H (\nabla du)$, $Q = 4n+6$, $2^* = 2Q/(Q-2)$.

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The qc-Yamabe problem: Find solutions to

$$4\,\frac{Q+2}{Q-2}\,\triangle u - \; u\,\mathrm{Scal} \;=\; -u^{2^*-1}\,\overline{\mathrm{Scal}}, \quad \overline{Scal} = const.$$

- Yamabe functional is

$$\Upsilon(u) = \int_M (4\,\frac{Q+2}{Q-2}\,|\nabla_H u|^2 \;+\; \mathrm{Scal}\,u^2)\,dv_g.$$

- The Yamabe invariant is the infimum

$$\Upsilon([\eta]) = \inf_u\{\Upsilon(u) : \;\int_M u^{2^*}\,dv_g \;=\; 1,\; u > 0\}.$$

Main strategy: Replace the non-linear Yamabe equation on $\boldsymbol{G}(\mathbb{H})$ with a geometrical system which can be solved. Use Cayley to transform the problem to the 3-Sasakian sphere and use compactness.

The first observation is:

Theorem (I/, I. Minchev, D. Vassilev)

If M is qc-Einstein then Scal=const.

Hint: Try to replace the Yamabe equation with conformal deformations preserving the qc-Einstein condition.

Let $\Psi \in$End(H).

- $Sp(n)$-invariant parts as follows

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Explicitly, $4\Psi^{+++} = \Psi - I_1 \Psi I_1 - I_2 \Psi I_2 - I_3 \Psi I_3$, etc.

- The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Using End$(H) \overset{g}{\cong} \Lambda^{1,1}$ the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of the operator

$$\Upsilon \ = \ I_1 \otimes I_1 \ + \ I_2 \otimes I_2 \ + \ I_3 \otimes I_3.$$

If $n = 1$ then the space of symmetric endomorphisms commuting with all $I_i, i = 1, 2, 3$ is 1-dimensional, i.e. the [3]-component of any symmetric endomorphism Ψ on H is proportional to the identity, $\Psi_{[3]} = \frac{tr\,(\Psi)}{4} Id_{|H}$.

The Torsion Tensor. $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$.

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Define $T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3 \in \Psi_{[-1]}$. *We have*
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3-Sasakian manifold: M^{4n+3} is 3-Sasaki if its cone is hyperkähler,
$C = M^{4n+3} \times \mathbb{R}^+$, $\quad g_{con} = t^2 g + dt^2$, $\quad Hol(g_{con}) \in Sp(n+1)$. If J_1, J_2, J_3) are the three comp-lex structures on C then $\xi_s = J_s \frac{\partial}{\partial t}$ are the Reeb vector fields of the qc-structure.
The torsion of Biquard connection vanishes, $T_\xi = 0$ and any 3-Sasakian manifold is qc-Einstein.

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Theorem (w/ I. Minchev, D. Vassilev)

a) *Suppose* $Scal \neq 0$. *The next conditions are equivalent:*

i) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold;*

ii) *M is locally 3-Sasakian: locally there exists a matrix* $\Psi \in \mathcal{C}^\infty(M : SO(3))$, *s.t.,* $(\frac{16n(n+2)}{Scal}\Psi \cdot \eta, Q)$ *is 3-Sasakian;*

iii) *The torsion of the Biquard connection is identically zero.*

The components of the torsion tensor transform according to the following formulas: if $\bar{\eta} = \frac{1}{2h}\eta$

- $\overline{T}^0(X, Y) \ = \ T^0(X, Y) \ + \ h^{-1}\,[\nabla dh]_{[sym][-1]}$, where the symmetric part is given by

$$[\nabla dh]_{[sym]}(X, Y) = \nabla dh(X, Y) + \sum_{s=1}^{3} dh(\xi_s)\,\omega_s(X, Y).$$

- $\bar{U}(X, Y) \ = \ U(X, Y) \ + \ (2h)^{-1}[\,\nabla dh - 2h^{-1} dh \otimes dh]_{[3][0]}$ or if $f = \frac{1}{2h}$, $\bar{\eta} = f\eta$, then

$$\bar{U}(X, Y) \ = \ U(X, Y) \ - \ (2f)^{-1}[\,\nabla df]_{[3][0]}.$$

Einstein deformations

Solution on $\boldsymbol{G\,(\mathbb{H})}$:

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Theorem (I/ I. Minchev, D. Vassilev)

Let $\Theta = \frac{1}{2h}\tilde{\Theta}$ be a conformal deformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. If Θ is also qc-Einstein, then up to a left translation the function h is given by

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\,(x^2 + y^2 + z^2)\right],$$

where c and ν are positive constants. All functions h of this form have this property.

Einstein deformations

Solution on $\boldsymbol{G}\,(\mathbb{H})$:

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Let $\Theta = \frac{1}{2h}\tilde{\Theta}$ be a conformal deformation of the standard qc-structure $\tilde{\Theta}$ on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$. If Θ is also qc-Einstein, then up to a left translation the function h is given by

$$h = c\left[\left(1 + \nu\,|q|^2\right)^2 + \nu^2\,(x^2 + y^2 + z^2)\right],$$

where c and ν are positive constants. All functions h of this form have this property.

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The problem is to show that these are all solutions to the Yamabe equation on $G\left(\mathbb{H}\right)$ - we need Obata type theorem Using Cayley we translate the problem to the sphere which is compact and can apply the horisontal divergence formula:

Proposition

Let $\left(M^{4n+3},\ \eta,\ g_H\right)$ be a compact closed manifold with a contact quaternionic structure and σ a horizontal 1-form, $\sigma \in \Lambda^1\left(H\right)$. Then we have

$$\int_M \left(\nabla^*\sigma\right)\eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_1^{2n} = 0,$$

where $\nabla^*\sigma = -(\nabla\sigma)(e_\alpha; e_\alpha)$ and $\{e_\alpha\}_\alpha$ is an ONB frame on H, $\alpha = 1, \ldots, 4n$.

Theorem (I/ I. Minchev, D. Vassilev)

Suppose (M^7, η) *is a quaternionic contact structure conformal to a 3-Sasakian structure* $(M^7, \bar{\eta})$, $\tilde{\eta} = \frac{1}{2h}\eta$. *If* $Scal_\eta = Scal_{\tilde{\eta}} = 16n(n+2)$, $f = \frac{1}{2} + h + \frac{1}{4}h^{-2}|\nabla h|^2$ *we have*

$$div\Big\{ fD + \sum_{s=1}^{3}\Big(dh(\xi_s)\, F_s + 4dh(\xi_s) I_s A_s - \frac{10}{3} dh(\xi_s)\, I_s A \Big)\Big\} = f|T^0|^2 + h\,\langle QV,\, V\rangle.$$

Here, Q *is a positive definite matrix,* $V = (D_1, D_2, D_3, A_1, A_2, A_3)$, $A_i = I_i[\xi_j, \xi_k]$, $A = A_1 + A_2 + A_3$.

$$D_1(X) = -h^{-1} T^{0^{+---}}(X, \nabla h),\ D_2(X) = -h^{-1} T^{0^{-++-}}(X, \nabla h),\ D_3(X) = -h^{-1} T^{0^{--++}}(X, \nabla h),$$

$$F_s(X) = -h^{-1}\, T^0(X, I_s \nabla h), \quad s = 1, 2, 3.$$

In particular, η *is again qc-Einstein.*

Theorem (I/ I. Minchev, D. Vassilev)

Let $\tilde{\eta} = \frac{1}{2h}\eta$, $\tilde{\eta}$ standard quaternionic contact structure on the quaternionic unit sphere S^7. If η has constant qc-scalar curvature, then up to a multiplicative constant η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorphism

$$\phi \in \mathit{Diff}(M), \quad \phi^* \tilde{\eta} = \mu \Psi \tilde{\eta}, \quad \Psi \in \mathcal{C}^\infty(M : SO(3)),$$

$$\eta = \phi^* \tilde{\eta}.$$

Furthermore, $\lambda(S^7) = \Upsilon(\tilde{\eta}) = 48\,(4\pi)^{1/5}$ and this minimum value is achieved only by $\tilde{\eta}$ and its images under conformal quaternionic contact automorphisms.

Sketch of Proof

The Cayley transform is a conformal quaternionic contact diffeomorphism between the quaternionic Heisenberg group with its standard quaternionic contact structure $\tilde{\Theta}$ and $S \setminus \{(-1,0)\}$ with its standard structure $\tilde{\eta}$. Hence, up to a constant multiplicative factor and a quaternionic contact automorphism the forms $\mathcal{C}_*\tilde{\eta}$ and $\tilde{\Theta}$ are conformal to each other. It follows that the same is true for $\mathcal{C}_*\eta$ and $\tilde{\Theta}$. In addition, $\tilde{\Theta}$ is qc-Einstein by definition, while η and hence also $\mathcal{C}_*\eta$ are qc-Einstein as follows from the above theorem. According to our previous Theorem, up to a multiplicative constant factor, the relation between the forms $\mathcal{C}_*\tilde{\eta}$ and $\mathcal{C}_*\eta$ are known, related by a translation or dilation on the Heisenberg group. Hence, we conclude that up to a multiplicative constant, η is obtained from $\tilde{\eta}$ by a conformal quaternionic contact automorpism. From the conformal properties of the Cayley transform it follows that the minimum $\lambda(S^{4n+3})$ is achieved by a smooth 3-contact form, which due to the Yamabe equation is of constant qc-scalar curvature.

Theorem (I/ I. Minchev, D. Vassilev)

Let $\eta = f\tilde{\eta}$ be a conformal deformation of the standard qc-structure $\tilde{\eta}$ on the quaternionic sphere S^{4n+3}, $n > 1$. Suppose η has constant qc-scalar curvature and

i) *the vertical space of η is integrable; or*
ii) *the function f is the real part of an anti-CRF function;*

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Theorem (I/ I. Minchev, D. Vassilev)

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Definition

A smooth $\mathbb{H}$-valued function $F : M \longrightarrow \mathbb{H}$, $\quad F = f + iw + ju + kv$, is said to be an anti-CRF function if the smooth real valued functions f, w, u, v satisfy

$$df \; = \; I_1 dw \; + \; I_2 du \; + \; I_3 dv \;\; mod \;\; \eta,$$

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Note: On a 3-Sasakian, $df = I_1 dw + I_2 du + I_3 dv \quad mod \quad \tilde{\eta}$ implies $[\tilde{\nabla} df]_{[3][0]} = 0$.
Recall, $U(X, Y) \;=\; \tilde{U}(X, Y) \;-\; (2f)^{-1}[\,\tilde{\nabla} df]_{[3][0]}$.. Hence $U = \tilde{U}$.

Lemma (I/ I. Minchev, D. Vassilev)

Let $(M, \bar{\eta})$ be a compact qc-Einstein manifold of dimension $(4n+3)$, $n > 1$. Let $\eta = \frac{1}{2h}\,\bar{\eta}$ be a conformal deformation with $Scal_\eta$ =const. Then any one of the following conditions implies that η is a qc-Einstein structure.

i) *the vertical space of η is integrable;*

ii) *the QC structure η is qc-pseudo Einstein, $U = 0$; ($\nabla^* U = 0$ is enough)*

ii) *the QC structure η has $\nabla^* T^0 = 0$.*

The Bianchi Identities

$$\sigma_{X,Y,Z}\Big\{R(X,Y,Z,V) - g((\nabla_X T)(Y,Z),V) - g(T(T_{X,Y},Z),V)\Big\} = 0$$
$$\sigma_{X,Y,Z}\Big\{g((\nabla_X R)(Y,Z)V,W) + g(R(T_{X,Y},Z)V,W)\Big\} = 0$$

Theorem (I/ I. Minchev, D. Vassilev)

The divergences of the curvature tensors satisfy the system $B\,b\ =\ 0$, *where*

$$\mathbf{B}\ =\ \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16n(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$$\mathbf{b}\ =\ \Big(\ \nabla^* T^o,\quad \nabla^* U,\quad A,\quad dScal,\quad Ric(\xi_j, I_j\,.)\ \Big)^t,\quad A = I_1[\xi_2,\xi_3] + I_2[\xi_3,\xi_1] + I_3[\xi_1,\xi_2].$$

$$[Ric_0]_{[-1]}(X,Y)\ =\ (2n+2)T^0(X,Y)\ =\ -(2n+2)h^{-1}[\nabla dh]_{[sym][-1]}(X,Y)$$
$$[Ric_0]_{[3]}(X,Y)\ =\ 2(2n+5)U(X,Y)\ =\ -(2n+5)h^{-1}[\nabla dh - 2h^{-1}dh\otimes dh]_{[3][0]}(X,Y).$$

$$\int_M h\,|\,[Ric_o]_{[-1]}\,|^2\,\eta\wedge\omega^{2n}\ =\ (2n+2)\int\langle[Ric_o]_{[-1]},\nabla dh]\rangle\,\eta\wedge\omega^{2n}$$
$$= (2n+2)\int_M\langle\nabla^*\ [Ric_o]_{[-1]},\nabla h]\rangle\,\eta\wedge\omega^{2n}\ =\ 0.$$

Back to the qc-Yamabe problem

- Yamabe functional: $\Upsilon(u) = \int_M (4\frac{Q+2}{Q-2}\,|\nabla_H u|^2 \;+\; \mathrm{Scal}\, u^2)\, dv_g$.
- The Yamabe invariant is the infimum $\Upsilon([\eta]) = \inf_u\{\Upsilon(u) : \;\int_M u^{2^*}\, dv_g \;=\; 1,\; u > 0\}$.

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a) $\Upsilon_M([\eta]) \leq \Upsilon_{S^{4n+3}}([\tilde{\eta}])$.

b) *If* $\Upsilon_M([\eta]) < \Upsilon_{S^{4n+3}}([\tilde{\eta}])$, *then the Yamabe problem has a solution.*

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Following the Riemannian and CR cases it remains to investigate

- when $\Upsilon_M([\eta]) = \Upsilon_{S^{4n+3}}([\tilde{\eta}])$ and show the existence of a solution to the qc-Yamabe problem - the most difficult part (may lead to 'quaternionic positive mass').

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Theorem (conjectured)

On a compact qc manifods $\Upsilon_M([\eta]) < \Upsilon_{S^{4n+3}}([\tilde\eta])$ *unless it is locally qc-conformal to* S^{4n+3}.

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- Expand $\Upsilon_M([\eta]$ using this tensor to show the conjectured result- WORK IN PROGRESS.
  - Construct suitable coordinates to express $\Upsilon_M([\eta])$ in terms of $\Upsilon_{S^{4n+3}}([\tilde\eta])$ and the norm $|W^{qc}|^2$ - the first part DONE by Christopher S. Kunkel, arXiv:0807.0465

We define a curvature type tensor W^{qc} on $\mathbb{H}$ depending only on the torsion T^0, U and the scalar curvature *Scal* and show

Theorem (I/ D. Vassilev)

a) The qc conformal curvature W^{qc} is invariant under quaternionic contact conformal transformations, i.e., if

$$\bar{\eta} = \phi \Psi \eta \quad \textit{then} \qquad W^{qc}_{\bar{\eta}} = \phi \, W^{qc}_{\eta},$$

for any smooth positive function ϕ and any $SO(3)$-matrix Ψ.
b) A qc structure on a (4n+3)-dimensional smooth manifold is locally quaternionic contact conformal to the standard flat qc structure on the quaternionic Heisenberg group $\boldsymbol{G}\,(\mathbb{H})$ if and only if the qc conformal curvature vanishes, $W^{qc} = 0$.

Conformal Flatness. Weyl, Cartan-Chern-Moser Type Theorem

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Our proof is similar to the classical approach used by H.Weyl and it is different than that used by Chern-Moser where the Cartan method of equivalence is applied.

Conformal Curvature. L-Schouten tensor. WR - "Weyl" tensor.

- "Schouten" tensor $L(X,Y) = \frac{1}{2}T^0(X,Y) + U(X,Y) + \frac{Scal}{32n(n+2)}\, g(X,Y)$.
- Conformal curvature

$$
\begin{aligned}
WR(X,Y,Z,V) &= R(X,Y,Z,V) + (g \owedge L)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_s L)(X,Y,Z,V) \\
&- \frac{1}{2}\sum_{(i,j,k)} \omega_i(X,Y)\Big[L(Z,I_iV) - L(I_iZ,V) + L(I_jZ,I_kV) - L(I_kZ,I_jV)\Big] \\
&- \sum_{s=1}^{3}\omega_s(Z,V)\Big[L(X,I_sY) - L(I_sX,Y)\Big] + \frac{1}{2n}(trL)\sum_{s=1}^{3}\omega_s(X,Y)\omega_s(Z,V),
\end{aligned}
$$

where $\owedge$ is the Kulkarni-Nomizu product of symmetric tensors and $\sum_{(i,j,k)}$ denotes the cyclic sum.

Proposition

a) *The [-1]-part w. r. t. the first two arguments of WR vanishes,*
$WR_{[-1]}(X, Y, Z, V) = \frac{1}{4}\Big[3WR(X, Y, Z, V) - \sum_{s=1}^{3} WR(I_sX, I_sY, Z, V)\Big] = 0.$

We define the qc-conformal curvature tensor $W^{qc} = WR_{[3]}$.

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b) *The [3]-part w. r. t. the first two arguments of WR is determined by the torsion and the scalar curvature*

$$\begin{aligned} WR_{[3]}(X,Y,Z,V) &= \frac{1}{4}\Big[R(X,Y,Z,V) + \sum_{s=1}^{3} R(I_sX, I_sY, Z, V)\Big] \\ &\quad - \frac{1}{2}\sum_{s=1}^{3}\omega_s(Z,V)\Big[T^0(X, I_sY) - T^0(I_sX, Y)\Big] \\ &\quad + \frac{Scal}{32n(n+2)}\Big[(g \owedge g)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge \omega_s)(X,Y,Z,V)\Big] \\ &\quad + (g \owedge U)(X,Y,Z,V) + \sum_{s=1}^{3}(\omega_s \owedge I_sU)(X,Y,Z,V). \end{aligned}$$

We define the qc-conformal curvature tensor $W^{qc} = WR_{[3]}$.

A consequence of the Bianchi identities:

Theorem (I/ Vasilev)

The following tensors

- $R(X,Y,Z,V) - R(Z,V,X,Y)$
- $4R_{[-1]}(X,Y,Z,V) =$ $3R(X,Y,Z,V) - R(I_1X, I_1Y, Z, V) - R(I_2X, I_2Y, Z, V) - R(I_3X, I_3Y, Z, V)$
- $R(\xi_i, X, Y, Z)$
- $R(\xi_i, \xi_j, X, Y)$

are determined by the (horizontal!) torsion tensor, i.e., T^0, U *and* $Scal$.

Corrolary

A QC manifold is locally isomorphic to the quaternionic Heisenberg group exactly when the curvature of the Biquard connection restricted to H vanishes, $R_{|_H} = 0$.

Sketch of the proof of the conformal flatness theorem

- Conformal invariance-long direct standard calculations and careful analysis of the structure of the qc-conformally related curvatures.

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Sketch of the proof of the conformal flatness theorem

- Conformal invariance-long direct standard calculations and careful analysis of the structure of the qc-conformally related curvatures.
- Suppose $W^{qc} = 0$. Then $WR = 0$ We look for a smooth function such that after a conformal transformation the new qc structure has flat Biquard connection restricted to the common horizontal space H.
- We consider the following overdetermined system of partial differential differential equations:

$$\nabla du(X,Y) = -du(X)du(Y) + du(I_1X)du(I_1Y) + du(I_2X)du(I_2Y) + du(I_3X)du(I_3Y)$$
$$+\frac{1}{2}g(X,Y)|du|^2 - du(\xi_1)\omega_1(X,Y) - du(\xi_2)\omega_2(X,Y) - du(\xi_3)\omega_3(X,Y) - L(X,Y), \quad (1)$$

$$\nabla du(X,\xi_i) = \mathbb{B}(X,\xi_i)$$
$$- L(X, I_i du) + \frac{1}{2}du(I_iX)|du|^2 - du(X)du(\xi_i) - du(I_jX)du(\xi_k) + du(I_kX)du(\xi_j), \quad (2)$$

$$\nabla du(\xi_1,\xi_1) = -\mathbb{B}(\xi_1,\xi_1) + \mathbb{B}(I_1du,\xi_1) + \frac{1}{4}|du|^4 - (du(\xi_1))^2 + (du(\xi_2))^2 + (du(\xi_3))^2, \quad (3)$$

$$\nabla du(\xi_2,\xi_1) = -\mathbb{B}(\xi_2,\xi_1) + \mathbb{B}(I_1du,\xi_2) - 2du(\xi_1)du(\xi_2) - \frac{Scal}{16n(n+2)}du(\xi_3), \quad (4)$$

$$\nabla du(\xi_3,\xi_1) = -\mathbb{B}(\xi_3,\xi_1) + \mathbb{B}(I_1du,\xi_3) - 2du(\xi_1)du(\xi_3) + \frac{Scal}{16n(n+2)}du(\xi_2). \quad (5)$$

where $\mathbb{B}(X,\xi_i)$ and $\mathbb{B}(\xi_i,\xi_j)$ do not depend on the unknown function u.

$$\mathbb{B}(X,\xi_i) = \frac{1}{2(2n+1)}\Big[(\nabla_{e_a}L)(I_ie_a,X) + \frac{1}{3}\Big((\nabla_{e_a}L)(e_a,I_iX) - \nabla_{I_iX}tr\,L\Big)\Big],$$
$$\mathbb{B}(\xi_s,\xi_t) = \frac{1}{4n}\Big[(\nabla_{e_a}\mathbb{B})(I_se_a,\xi_t) + L(e_a,e_b)L(I_te_a,I_se_b)\Big].$$

The integrability conditions for this over-determined system are the Ricci identities for the Biquard connection.

$$\nabla du(A,B,C) - \nabla du(B,A,C) = -R(A,B,C,du) - \nabla du((T(A,B),C), \quad A,B,C \in \Gamma(TM).$$

After very long calculations we show that these conditions are consequence of $W^{qc} = 0$ applying the Bianchi identities for the Biquard connection.

Integrability conditions read:

$$(\nabla_Z L)(X,Y)-(\nabla_X L)(Z,Y)=\sum_{s=1}^{3}\Big[\omega_s(Z,Y)\mathbb{B}(X,\xi_s)-\omega_s(X,Y)\mathbb{B}(Z,\xi_s)+2\omega_s(Z,X)\mathbb{B}(Y,\xi_s)\Big].$$

$$(\nabla_{\xi_t} L)(X,Y)+(\nabla_X\mathbb{B})(Y,\xi_t)+L(Y,I_tL(X))+L(T(\xi_t,X),Y)+g(T(\xi_t,Y),L(X))$$
$$=\sum_{s=1}^{3}\mathbb{B}(\xi_s,\xi_t)\omega_s(X,Y),\quad t=1,2,3.$$

$$(\nabla_{\xi_1}\mathbb{B})(X,\xi_2)+(\nabla_X\mathbb{B})(\xi_1,\xi_2)-2L(X,I_2e_a)\mathbb{B}(e_a,\xi_1)+T(\xi_1,X,e_a)\mathbb{B}(e_a,\xi_2)-\frac{1}{2n}trL\,\mathbb{B}(X,\xi_3)=0.$$

$$(\nabla_{\xi_2}\mathbb{B})(X,\xi_2)\;+\;(\nabla_X B)(\xi_2,\xi_2)\;-\;2\mathbb{B}(\mathbf{e}_a,\xi_2)L(X,I_2e_a)\;+\;T(\xi_2,X,e_a)\mathbb{B}(e_a,\xi_2)\quad=\quad 0.$$

$$\nabla_{\xi_1}\mathbb{B}(\xi_3,\xi_2)-\nabla_{\xi_3}\mathbb{B}(\xi_1,\xi_2)=\frac{1}{2n}(tr\,L)\,[\mathbb{B}(\xi_1,\xi_1)-2\mathbb{B}(\xi_2,\xi_2)+\mathbb{B}(\xi_3,\xi_3)]$$
$$+2\mathbb{B}(e_a,\xi_1)\mathbb{B}(I_2e_a,\xi_3)+\mathbb{B}(e_a,\xi_1)\mathbb{B}(I_3e_a,\xi_2)+\mathbb{B}(I_1e_a,\xi_3)\mathbb{B}(e_a,\xi_2).$$

$$\nabla_{\xi_2}\mathbb{B}(\xi_3,\xi_2)-\nabla_{\xi_3}\mathbb{B}(\xi_2,\xi_2)=-B(I_3e_a,\xi_2)\mathbb{B}(e_a,\xi_2)+3\mathbb{B}(I_2e_a,\xi_3)\mathbb{B}(e_a,\xi_2)+\frac{3}{n}(tr\,L)\,\mathbb{B}(\xi_1,\xi_2).$$

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Theorem (I/ D. Vassilev)

Let (M, η) be a compact quaternionic contact manifold and G a connected Lie group of conformal quaternionic contact automorphisms of M. If G is non-compact then M is qc conformally equivalent to the unit sphere S in quaternionic space.