Lie n-algebras and supersymmetry

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in collaboration with:

Paul de Medeiros + Elena Méndez-Escobar

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- Metric Lie algebras
- Some structure theory
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Introduction

What is a Lie n-algebra?

One of several possible n-ary generalisations of a **Lie algebra**, which is the case $n=2$.

A Lie algebra is a vector space V together with a bilinear bracket $[-,-]: V \times V \to V$ satisfying:

alternating $[x, x] = 0$

Jacobi identity $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$

How do we generalise this to an n-ary bracket?

alternating $[x_1, x_2, \ldots, x_n] = 0 \quad \text{if } x_i = x_j \ \exists\, i \neq j$

What about the **Jacobi identity**?

$[x_1, \ldots, x_{n-1}, -] : V \to V$ is a derivation.

Filippov (1985)

Also satisfied by Nambu-Poisson bracket.

Nambu (1973)

Other generalisations are possible, but they do not seem to arise in “nature”.

Lie n-algebras in "nature"

- Nambu mechanics

Nambu (1973), Takhtajan (1994)

- Supersymmetric supergravity backgrounds

FO+Papadopoulos (2002)

- Supersymmetric gauge theories

Bagger+Lambert (2006,2007), Gustavsson (2007)

Metric Lie n-algebras

Most naturally occurring Lie n-algebras are **metric**; that is, they leave invariant a nondegenerate symmetric bilinear form:

$$\langle [x_1, \ldots, x_{n-1}, y], z \rangle = - \langle [x_1, \ldots, x_{n-1}, z], y \rangle$$

Metric Lie n-algebras are closed under *orthogonal direct sum* and hence there is a notion of **indecomposable** metric Lie n-algebra.

The n-Jacobi identity for a metric Lie n-algebra is reminiscent of a Plücker formula.

Let F be the $(n+1)$-form

$$F(x_1, \ldots, x_{n+1}) = \langle [x_1, \ldots, x_n], x_{n+1} \rangle$$

so that

$$F(x_1, \ldots, x_{n-1}, -, -) \in \Lambda^2 V \cong \mathfrak{so}(V)$$

Then the n-Jacobi identity becomes

$$\imath_\Xi F \cdot F = 0 \qquad \forall\ \Xi \in \Lambda^{n-1} V$$

This identity (for $n=4$) is implied by the flatness of the “gravitino” connection on the spin bundle of a type IIB supergravity background.

FO+Papadopoulos (2002)

Based on explicit low-dimensional (and hence low-n) calculations, we conjectured that there is up to isomorphism a unique nonabelian indecomposable **positive-definite** metric Lie n-algebra:

$e_1, \ldots, e_{n+1}$ orthonormal basis for V

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = (-1)^i e_i$$

This is now known to be true.

Nagy (2007)

Papadopoulos, Gauntlett+Gutowski (2008) for $n=3$

Papadopoulos (2008)

(We also made a lorentzian conjecture which is now known to be false.)

Why the recent interest?

AdS/CFT

There is now a proposal for the superconformal field theory dual to multiple **M2**-branes, whose existence is predicted by **AdS/CFT**.

Bagger+Lambert (2006,2007), Gustavsson (2007)

Recall that AdS/CFT predicts a duality between string/M-theory on the near-horizon geometry of a non-dilatonic brane (i.e., **M2**,**M5**,**D3**) and a superconformal field theory on its conformal boundary.

Maldacena (1997), Gubser+Klebanov+Polyakov (1998), Witten(1998)

The best-understood instance of AdS/CFT is the duality between **type IIB string theory** on the near-horizon geometry of N coincident **D3**-branes:

$$\mathrm{AdS}_5 \times S^5 \quad \text{with equal radii of curvature} \propto N^{1/4}$$

and the maximally **supersymmetric** $SU(N)$ **Yang-Mills theory** on the conformal compactification of Minkowski spacetime.

Both theories admit actions of isomorphic Lie superalgebras as supersymmetries.

The **Killing superalgebra** of the near-horizon supergravity background

$$\mathrm{AdS}_5 \times S^5$$

is isomorphic to the **conformal superalgebra** of the supersymmetric Yang-Mills theory: $\mathfrak{su}(2,2|4)$.

The even subalgebra is $\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$.

It is the **isometry** Lie algebra of the near-horizon geometry and also the **conformal + R-symmetry** of the conformal field theory.

M2-branes

Eleven dimensional supergravity admits a two-parameter family of half-supersymmetric backgrounds:

$$\begin{aligned} g &= H^{-2/3} g\left(\mathbb{R}^{1,2}\right) + H^{1/3}\left(dr^2 + r^2 g\left(S^7\right)\right) \\ F &= \mathrm{dvol}\left(\mathbb{R}^{1,2}\right) \wedge dH^{-1} \end{aligned}$$

where

$$H = \alpha + \frac{\beta}{r^6}$$

For generic α and $\beta \propto N$, this describes N coincident **M2**-branes.

For $\beta = 0$, the background becomes (*11*-dimensional) Minkowski spacetime, whereas for $\alpha = 0$, it becomes

$$\mathrm{AdS}_4 \times S^7 \quad \text{with} \quad R_{\mathrm{AdS}} = \tfrac{1}{2} R_S \propto \beta^{1/6}$$

which is the near-horizon geometry of the N coincident **M2**-branes.

AdS/CFT predicts the existence of a three-dimensional supersymmetric gauge theory with superconformal algebra isomorphic to the Killing superalgebra of the near-horizon geometry of the **M2**-branes: $\mathfrak{osp}(8|2)$.

The even subalgebra is isomorphic to

$$\mathfrak{so}(2,3) \oplus \mathfrak{so}(8)$$

conformal algebra of $\mathbb{R}^{1,2}$

R-symmetry

The construction of such a theory remained elusive until 2007, when a proposal emerged, one of whose essential ingredients is a metric Lie 3-algebra.

Bagger+Lambert, Gustavsson (2007)

Bagger-Lambert Theory

V	**metric Lie 3-algebra**
E	vector rep of ***SO(8)***
Σ	irrep spinor of ***Spin(1,2)***
S	+ve chirality spinor irrep of ***Spin(8)***

(We use the same names for the corresponding trivial bundles.)

The field content consists of:

a scalar field $X \in C^\infty(\mathbb{R}^{1,2}; V \otimes E)$

a "gauge" field $A \in \Omega^1(\mathbb{R}^{1,2}; \Lambda^2 V)$

a fermion field $\psi \in C^\infty(\mathbb{R}^{1,2}; V \otimes \Sigma \otimes S)$

The Lie 3-algebra structure on V defines a map

$$\mathrm{ad} : \Lambda^2 V \to \mathfrak{so}(V) \qquad \mathrm{ad}(x \wedge y) = [x, y, -]$$

whose image is the Lie algebra of **inner derivations** of V.

The gauge field defines a connection on V:

$$D_A = d + \mathrm{ad}(A)$$

which can be twisted by the spin connection to define a twisted Dirac operator:

$$\mathcal{D}_A \psi = \mathcal{D}\psi + \mathrm{ad}(A) \cdot \psi$$

The lagrangian is given by:

$$\mathcal{L} = \left(\tfrac{i}{2}\langle\psi, \mathcal{D}_A\psi\rangle - \tfrac{1}{2}|D_A X|^2 - i\langle\psi, [X^2]\cdot\psi\rangle - \tfrac{1}{2}|[X^3]|^2\right)\mathrm{dvol}(\mathbb{R}^{1,2})$$
$$+ \tfrac{1}{2}[A \wedge dA] + \tfrac{1}{3}[\![A^3]\!]$$

where

$$X^2 \in \mathfrak{so}(E) \otimes \Lambda^2 V \implies [X^2] \in \mathfrak{so}(E) \otimes \mathfrak{so}(V)$$

$$X^3 \in \Lambda^3 E \otimes \Lambda^3 V \implies [X^3] \in \Lambda^3 E \otimes V$$

$$A \wedge dA \in \Omega^3(\mathbb{R}^{1,2}; \Lambda^2 V \otimes \Lambda^2 V) \implies [A \wedge dA] \in \Omega^3(\mathbb{R}^{1,2})$$

$$A^3 \in \Omega^3(\mathbb{R}^{1,2}; \Lambda^2 V \otimes \Lambda^2 V \otimes \Lambda^2 V) \implies [\![A^3]\!] \in \Omega^3(\mathbb{R}^{1,2})$$

The equations of motion only depend on the gauge field via *ad(A)* and $\mathcal{L}$ is gauge-invariant under ***Ad(V)***, the Lie subgroup of ***SO(V)*** generated by the inner derivations of ***V***.

$\mathcal{L}$ is also invariant under ***N=8*** supersymmetry transformations, which again depend on *ad(A)*. It is in fact the **closure** of the supersymmetry algebra which imposes that ***V*** be a Lie 3-algebra.

Recently, similar actions with less supersymmetry have been proposed which require ***V*** to be other **triple systems**.

Aharony+Bergman+Jafferis+Maldacena, Bagger+Lambert, Schnabl+Tachikawa, Cherkis+Sämann (2008)

To every metric Lie 3-algebra V, there is associated a Bagger-Lambert model.

This makes their classification into an interesting problem.

In addition, many physical properties of the Bagger-Lambert model can be rephrased as properties of the metric Lie 3-algebra.

This refines further the classification problem.

One is interested in classifying **indecomposable metric Lie 3-algebras** admitting the following:

- a **maximally isotropic centre**, for decoupling of negative-norm states;
- a **conformal automorphism**, for scale invariance of the theory; and
- an **isometric anti-automorphism**, for parity invariance of the theory.

de Medeiros+FO+Méndez-Escobar (2008)

Metric Lie algebras

Recall that a real Lie algebra is **metric** if it admits an ad-invariant inner product; equivalently, if its *1*-connected Lie group has a **bi-invariant** metric.

There is no classification (except for small index), but there is a structure theorem.

Abelian Lie algebras are metric, since ad-invariance is vacuous.

Simple Lie algebras are metric relative to the Killing form.

Therefore so are **reductive** Lie algebras, by taking (orthogonal) direct sums.

However there are also **non-reductive** metric Lie algebras, e.g.,

$$\mathfrak{g} \ltimes \mathfrak{g}^*$$

relative to the canonical dual pairing.

The general metric Lie algebra is obtained by a mixture of these two operations.

Double extensions

$\mathfrak{g}$ a metric Lie algebra

$\mathfrak{h}$ a Lie algebra

$\varphi : \mathfrak{h} \to \mathfrak{so}(\mathfrak{g})$ via isometric derivations

$\mathfrak{D}(\mathfrak{g};\mathfrak{h}) = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ is a metric Lie algebra w.r.t.

$$[(X,h,\alpha),(Y,k,\beta)] = \\ ([X,Y] + \varphi(h)Y - \varphi(k)X, [h,k], \mathrm{ad}_h^*\beta - \mathrm{ad}_k^*\alpha + \varphi^*(X,Y))$$

$$\langle (X,h,\alpha),(Y,k,\beta)\rangle = \langle X,Y\rangle + \langle h,k\rangle + \alpha(k) + \beta(h)$$

Theorem. An indecomposable metric Lie algebra is either one-dimensional, simple or else it is a double extension of a metric Lie algebra by a one-dimensional or simple Lie algebra.

Theorem. The class of metric Lie algebras is generated by the one-dimensional and simple Lie algebras by iterating the operations of orthogonal direct sum and double extension.

Medina+Revoy (1985)

Double-extending increases the index:

$$\operatorname{index} \mathfrak{D}(\mathfrak{g}; \mathfrak{h}) = \operatorname{index} \mathfrak{g} + \dim \mathfrak{h}$$

Corollary. A positive-definite metric Lie algebra is reductive.

Simple algebras factor out:

$$\mathfrak{D}(\mathfrak{g} \oplus \mathfrak{s}; \mathfrak{h}) \cong \mathfrak{D}(\mathfrak{g}; \mathfrak{h}) \oplus \mathfrak{s} \qquad \text{for } \mathfrak{s} \text{ simple}$$

FO+Stanciu (1994)

Corollary. An indecomposable lorentzian Lie algebra is either one-dimensional, simple, or the double extension of an abelian Lie algebra by a one-dimensional Lie algebra.

It follows that an indecomposable lorentzian Lie algebra is isomorphic to one of the following:

- one-dimensional
- ***so(1,2)***
- ***E⊕R(u,v)***

$$[u, x] = A(x) \qquad [x, y] = \langle A(x), y \rangle \, v$$

$$A \in \mathfrak{so}(E) \qquad \langle u, v \rangle = 1 \quad \langle v, v \rangle = 0$$

Medina (1985)

For index ≥ 2, the double extension can be ambiguous and a different method based on quadratic cohomology has been developed, resulting in a classification for index 2.

Baum+Kath, Kath+Olbrich (2002)

Some structure theory

There is a reasonably developed structure theory for Lie n-algebras. For $n=3$ it parallels the structure theory of **Lie triple systems**. In general, it is governed largely by the theory of Lie algebras, applied to the Lie algebra of inner derivations.

Filippov (1985), Kasymov (1987), Ling (1993)

Many of the results for metric Lie n-algebras parallel those for metric Lie algebras and some theorems have almost identical statements.

Basic definitions

Let V denote a Lie 3-algebra. (All this works for Lie n-algebras with the obvious changes.)

$W \subset V$ is a **subalgebra** if $[W,W,W] \subset W$.

$I \subset V$ is an **ideal** if $[I,V,V] \subset I$.

(This guarantees a one-to-one correspondence between ideals and kernels of homomorphisms.)

V is **simple** if it is not one-dimensional and does not possess any proper ideals.

Theorem. There is a unique complex simple Lie 3-algebra up to isomorphism. It is 4-dimensional and there is a basis relative to which

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_4] = (-1)^i e_i$$

Over the reals, there are precisely three simple Lie 3-algebras, up to isomorphism:

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_4] = (-1)^i \varepsilon_i e_i$$

where the ϵ's are signs: $(++++)$, $(+++-)$, $(++--)$.

Ling (1993)

They are metric of index 0,1 and 2, respectively.

There is a notion of **double extension** for Lie n-algebras, but it is not as easy to describe. Take $n=3$, to illustrate:

$\mathfrak{W}$ a metric Lie 3-algebra

$\mathfrak{V}$ a Lie 3-algebra

$\mathfrak{D}(\mathfrak{W};\mathfrak{V}) = \mathfrak{W} \oplus \mathfrak{V} \oplus \mathfrak{V}^*$ a metric Lie 3-algebra

$$[\mathfrak{V}\,\mathfrak{V}\,\mathfrak{V}] \subset \mathfrak{V} \qquad [\mathfrak{W}\,\mathfrak{W}\,\mathfrak{W}] \subset \mathfrak{W} \oplus \mathfrak{V}^*$$

$$[\mathfrak{V}\,\mathfrak{V}\,\mathfrak{V}^*] \subset \mathfrak{V}^* \qquad [\mathfrak{V}\,\mathfrak{W}\,\mathfrak{W}] \subset \mathfrak{W} \oplus \mathfrak{V}^*$$

$$[\mathfrak{V}\,\mathfrak{V}\,\mathfrak{W}] \subset \mathfrak{W}$$

subject to the 3-Jacobi identity.

de Medeiros+FO+Méndez-Escobar, FO (2008)

Theorem. An indecomposable metric Lie 3-algebra is either one-dimensional, simple or else it is a double extension of a metric Lie 3-algebra by a one-dimensional or simple Lie 3-algebra.

Theorem. The class of metric Lie 3-algebras is generated by the one-dimensional and simple Lie 3-algebras by iterating the operations of orthogonal direct sum and double extension.

de Medeiros+FO+Méndez-Escobar (2008)

The same holds for Lie n-algebras for $n>3$.

FO (2008)

Classifications

Theorem. An indecomposable lorentzian Lie 3-algebra is either one-dimensional, simple or else isomorphic to

$$E \oplus \mathbb{R}u \oplus \mathbb{R}v$$

$$[u, x, y] = [x, y] \qquad [x, y, z] = -\langle [x, y], z \rangle\, v$$

where $\quad x, y, z \in E \qquad \langle u, v \rangle = 1 \quad \langle v, v \rangle = 0$

and $\qquad (E, [-, -], \langle -, - \rangle)$

is a compact semisimple Lie algebra with a choice of ad-invariant inner product.

de Medeiros+FO+Méndez-Escobar (2008)

Theorem. An indecomposable lorentzian Lie n-algebra is either one-dimensional, simple or else isomorphic to $\mathfrak{W} \oplus \mathbb{R}u \oplus \mathbb{R}v$

$$[u, x_1, \ldots, x_{n-1}] = [x_1, \ldots, x_{n-1}]$$
$$[x_1, \ldots, x_n] = (-1)^n \langle [x_1, \ldots, x_{n-1}], x_n \rangle v$$

where $x_i \in \mathfrak{W}$ $\langle u, v \rangle = 1$ $\langle v, v \rangle = 0$

and $(\mathfrak{W}, [-, \ldots, -], \langle -, - \rangle)$

is a semisimple euclidean Lie *(n-1)*-algebra with a choice of invariant inner product.

FO (2008)

We have also classified metric Lie 3-algebras of index 2. There are some 10 non-isomorphic classes of such Lie 3-algebras, of which 4 satisfy the physical criteria coming from Bagger-Lambert theory.

As they are rather involved, we will not describe them here, except to say that they are built out of compact semisimple Lie algebras and euclidean vector spaces with compatible symplectic structures.

Open questions

- ★ Is there a notion of Lie n-group?
- ★ Unpack the definition of a double extension of a metric Lie n-algebra. What is it ***really***?
- ★ Extension to other **triple systems**.
- ★ Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{p}\subset\mathfrak{g}$ be a *subspace*. If $[\mathfrak{p},\mathfrak{p}]=\mathfrak{p}^\perp$, is $\mathfrak{p}^\perp$ a Lie subalgebra of $\mathfrak{g}$?