

Harmonic almost contact structures via the intrinsic torsion

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For an oriented Riemannian manifold M of dimension n , given G a Lie subgroup of $SO(n)$, M is said to be equipped with a **G -structure**, if there exists a subbundle $\mathcal{G}(M)$, with structure group G , of the oriented orthonormal frame bundle $S\mathcal{O}(M)$.

For an oriented Riemannian manifold M of dimension n , given G a Lie subgroup of $SO(n)$, M is said to be equipped with a G -structure, if there exists a subbundle $\mathcal{G}(M)$, with structure group G , of the oriented orthonormal frame bundle $S\mathcal{O}(M)$.

G	$\dim M$	name of the G -structure
$U(n)$	$2n$	almost Hermitian
$SU(n)$	$2n$	special almost Hermitian
$U(n) \times 1$	$2n + 1$	almost contact metric
$Sp(n)$	$4n$	almost hyperHermitian
$Sp(n) Sp(1)$	$4n$	almost quaternion Hermitian
G_2	7	G_2 -structure
$Spin(7)$	8	$Spin(7)$ -structure

For a fixed G ,

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'which are the best G -structures on a given Riemannian manifold $(M, \langle \cdot, \cdot \rangle_M)$?'

$$(M, \langle \cdot, \cdot \rangle_M), \quad (N, \langle \cdot, \cdot \rangle_N)$$

$$f : M \rightarrow N$$

For M compact and oriented, the **energy** of f is given by:

$$\mathcal{E}(f) = \frac{1}{2} \int_M \|f_*\|^2 dv$$

$$\|f_*\|^2 = \langle f_* e_i, f_* e_i \rangle_N$$

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Tension field

$$\tau(f) = \tilde{\nabla}_{e_i}(f_*e_i) - f_*\nabla_{e_i}e_i,$$

where $\tilde{\nabla}$ is the induced connection by ∇^N on f^*TN

the pullback bundle $f^*TN = \{(m, \tilde{X}), \quad m \in M \text{ and } \tilde{X} \in T_{f(m)}N\}$

f harmonic map if and only if $\tau(f) = 0$

J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*,
Amer. J. Math. 86 (1964), 109-160.

$(M, \langle \cdot, \cdot \rangle)$ compact and oriented, $G \subseteq SO(n)$, G closed and connected

$$\mathcal{G}(M) \subseteq S\mathcal{O}(M)$$

The presence of a G -structure is equivalent to the presence of a section

$$\sigma : M \rightarrow S\mathcal{O}(M)/G$$

$$\langle A, B \rangle_{S\mathcal{O}(M)/G} = \langle \pi_* A, \pi_* B \rangle + \langle \phi A, \phi B \rangle.$$

The **energy** of a G -structure σ

$$\mathcal{E}(\sigma) = \frac{1}{2} \int_M \|\sigma_*\|^2 dv$$

C. M. Wood, Harmonic sections of homogeneous fibre bundles, *Differential Geom. Appl.* 19 (2003), 193-210

J. C. González-Dávila and FMC, Harmonic G -structures, *Math. Proc. Cambridge Philos. Soc.* (to appear). arXiv:math.DG/0706.0116

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m}, \quad \langle \varphi_1, \varphi_2 \rangle = \langle \varphi_1(u_i), \varphi_2(u_i) \rangle$$
$$(g \cdot \varphi)(x) = g\varphi(g^{-1}x), \quad x \in \mathbb{R}^n$$

$$\mathfrak{so}(M) = \mathfrak{g}_\sigma \oplus \mathfrak{m}_\sigma,$$

G-connection: $\tilde{\nabla}$, **torsion:** $\tilde{\xi}_X = \tilde{\nabla}_X - \nabla_X \in \mathfrak{so}(M)$

$$\tilde{\xi}_X = (\tilde{\xi}_X)_{\mathfrak{g}_\sigma} + (\tilde{\xi}_X)_{\mathfrak{m}_\sigma}$$

minimal connection of σ : $\nabla_X^G = \tilde{\nabla}_X - (\tilde{\xi}_X)_{\mathfrak{g}_\sigma}$

intrinsic torsion of σ : $\xi_X^G = (\tilde{\xi}_X)_{\mathfrak{m}_\sigma} = \nabla_X^G - \nabla_X \in \mathfrak{m}_\sigma$

For a G -structure σ :

minimal connection ∇^G , intrinsic torsion $\xi^G \in T^*M \otimes \mathfrak{m}_\sigma$

$$\nabla^G = \nabla + \xi^G,$$

S. Salamon, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Math. Series, **201**, Longman (1989).

R. L. Bryant, Metrics with exceptional holonomy, *Ann. of Math.* **126** (1987), 525–576.

R. Cleyton and A. F. Swann, Einstein metrics via intrinsic or parallel torsion, *Math. Z.* 247 no. 3(2004), 513–528.

$SO(M)/G$ as a Riemannian manifold

$$SO(M) \xrightarrow{\pi_{SO(n)}} M \qquad SO(M) \xrightarrow{\pi_G} SO(M)/G$$

$$TSO(M)/G = \mathcal{V} \oplus \mathcal{H}$$

$$\mathcal{V} = \pi_{G*}(\ker \pi_{SO(n)*}) \qquad \mathcal{H} = \pi_{G*}(\ker \omega)$$

$\omega : TSO(M) \rightarrow \mathfrak{so}(n)$ is the connection one-form of the Levi Civita connection ∇

$$S\mathcal{O}(M)/G \xrightarrow{\pi} M$$

$$\pi^* \mathfrak{so}(M) = S\mathcal{O}(M) \times_G \mathfrak{so}(n) = \mathfrak{g}_{S\mathcal{O}(M)} \oplus \mathfrak{m}_{S\mathcal{O}(M)},$$

where $\mathfrak{g}_{S\mathcal{O}(M)} = S\mathcal{O}(M) \times_G \mathfrak{g}$ and $\mathfrak{m}_{S\mathcal{O}(M)} = S\mathcal{O}(M) \times_G \mathfrak{m}$.

A fibred metric on $\pi^* \mathfrak{so}(M)$ is defined by

$$\langle (pG, \varphi_m), (pG, \psi_m) \rangle = \langle \varphi_m, \psi_m \rangle$$

$$\phi|_{\mathcal{V}_{pG}} : \mathcal{V}_{pG} \rightarrow (\mathfrak{m}_{\mathcal{S}\mathcal{O}(M)})_{pG}$$

Any vector in \mathcal{V}_{pG} is given by $\pi_{G*p}(a_p^*)$, for some $a = (a_{ji}) \in \mathfrak{m}$

$$\phi|_{\mathcal{V}_{pG}}(\pi_{G*p}(a_p^*)) = (pG, a_{ji} p(u_i))^b \otimes p(u_j)$$

p is an orthonormal frame on $m \in M$, $p : \mathbb{R}^n \rightarrow T_m M$, and $u_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$.

Extending ϕ to $T_{pG}\mathcal{S}\mathcal{O}(M)/G$, by saying $\phi|_{\mathcal{H}_{pG}} = 0$, one can define

$$\langle A, B \rangle_{\mathcal{S}\mathcal{O}(M)/G} = \langle \pi_* A, \pi_* B \rangle + \langle \phi A, \phi B \rangle$$

$\pi : \mathcal{S}\mathcal{O}(M)/G \rightarrow M$ is a Riemannian submersion with totally geodesic fibres [Vilms] (Besse's book)

$$\langle A, B \rangle_{\mathcal{SO}(M)/G} = \langle \pi_* A, \pi_* B \rangle + \langle \phi A, \phi B \rangle$$

$\pi : \mathcal{SO}(M)/G \rightarrow M$ is a Riemannian submersion with totally geodesic fibres

$$\mathcal{E}(\sigma) = \frac{1}{2} \int_M \|\sigma_*\|^2 dv = \frac{n}{2} \text{Vol}(M) + \frac{1}{2} \int_M \|\phi \sigma_*\|^2 dv$$

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$$\phi \sigma_* = -\xi^G$$

Total bending

$$B(\sigma) = \frac{1}{2} \int_M \|\phi \sigma_*\|^2 dv = \frac{1}{2} \int_M \|\xi^G\|^2 dv$$

$t \rightarrow \sigma_t \in \Gamma^\infty(\mathcal{S}\mathcal{O}(M)/G)$ variation such that $\sigma_0 = \sigma$
 variation field $m \rightarrow \varphi(m) = \frac{d}{dt}|_{t=0} \sigma_t(m)$, $\varphi \in \Gamma^\infty(\sigma^*\mathcal{V})$.
 Therefore, $\Gamma^\infty(\sigma^*\mathcal{V}) \cong T_\sigma \Gamma^\infty(\mathcal{S}\mathcal{O}(M)/G)$

$$\sigma^*\mathcal{V} \cong \sigma^*\mathfrak{m}_{\mathcal{S}\mathcal{O}(M)} \cong \mathfrak{m}_\sigma$$

Then $T_\sigma \Gamma^\infty(\mathcal{S}\mathcal{O}(M)/G) \cong \Gamma^\infty(\mathfrak{m}_\sigma)$

The first variation formula

If σ is a G-structure, then, for all
 $\varphi \in \Gamma^\infty(\mathfrak{m}_\sigma) \cong T_\sigma \Gamma^\infty(\mathcal{S}\mathcal{O}(M)/G)$, we have

$$d\mathcal{E}_\sigma(\varphi) = - \int_M \langle \xi^G, \nabla \varphi \rangle dv = - \int_M \langle d^* \xi^G, \varphi \rangle dv,$$

where ξ^G is the intrinsic torsion of σ .

Harmonic G-structures

The coderivative $d^*\xi^G$ is a global section of \mathfrak{m}_σ and is given by

$$d^*\xi^G = -(\nabla_{e_i}\xi^G)_{e_i} = -(\nabla_{e_i}^G\xi^G)_{e_i} - \xi_{\xi_{e_i}^G}^G.$$

the following conditions are equivalent:

- (i) σ is a critical point for the energy functional on $\Gamma^\infty(\mathcal{SO}(M)/G)$.
- (ii) $d^*\xi^G = 0$.
- (iii) $(\nabla_{e_i}^G\xi^G)_{e_i} = -\xi_{\xi_{e_i}^G}^G$.

$$(\nabla_X \sigma_*)(Y) = \nabla_X^q \sigma_* Y - \sigma_*(\nabla_X Y),$$

If σ is a G-structure on $(M, \langle \cdot, \cdot \rangle)$, then:

- (a) $\phi(\nabla_X \sigma_*)Y = -\frac{1}{2}((\nabla_X \xi^G)_Y + (\nabla_Y \xi^G)_X)$.
- (b) $2\langle \pi_*(\nabla_X \sigma_*)Y, Z \rangle = \langle \xi_X^G, R(Y, Z) \rangle + \langle \xi_Y^G, R(X, Z) \rangle$.

$$\tau(\sigma) = (\nabla_{e_i} \sigma_*)(e_i), \quad \pi_* \tau(\sigma) = \langle \xi_{e_i}^G, R(e_i, \cdot) \rangle^\sharp, \quad \phi \tau(\sigma) = d^* \xi^G$$

A G-structure σ on a closed and oriented Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is harmonic as a map if and only if σ is a harmonic G-structure such that

$$\langle \xi_{e_i}^G, R(e_i, \cdot) \rangle = 0.$$

$$\nabla^* \nabla \Psi = - (\nabla^2 \Psi)_{e_i, e_i}, \quad (\nabla^2 \Psi)_{X, Y} = \nabla_X (\nabla_Y \Psi) - \nabla_{\nabla_X Y} \Psi$$

Let $(M, \langle \cdot, \cdot \rangle)$ be an oriented Riemannian n -manifold equipped with a G -structure, where the Lie group G is closed, connected and $G \subseteq SO(n)$. If Ψ is a (r, s) -tensor field on M which is stabilised under the action of G , then

$$\nabla^* \nabla \Psi = (\nabla_{e_i}^G \xi^G)_{e_i} \Psi + \xi_{\xi_{e_i}^G}^G \Psi - \xi_{e_i}^G (\xi_{e_i}^G \Psi).$$

Moreover, if the G -structure is harmonic, then

$$\nabla^* \nabla \Psi = -\xi_{e_i}^G (\xi_{e_i}^G \Psi).$$

An almost contact metric manifold $(M^{2n+1}, \langle \cdot, \cdot \rangle, \varphi, \zeta)$

$$\begin{aligned}\langle \varphi X, \varphi Y \rangle &= \langle X, Y \rangle - \eta(X)\eta(Y) \\ \varphi^2 &= -I + \eta \otimes \zeta, \quad \zeta^b = \eta\end{aligned}$$

$$G = U(n) \times 1 \subseteq SO(2n+1), \quad T_m^* M = \mathbb{R}\eta + \eta^\perp$$

$$\mathfrak{so}(2n+1) \cong \Lambda^2 T^* M \cong \Lambda^2 \eta^\perp + \eta^\perp \wedge \mathbb{R}\eta = \mathfrak{u}(n) + \mathfrak{u}(n)_{|\zeta^\perp}^\perp + \eta^\perp \wedge \mathbb{R}\eta$$

$$\mathfrak{u}(n)^\perp = \mathfrak{u}(n)_{|\zeta^\perp}^\perp + \eta^\perp \wedge \mathbb{R}\eta$$

$$T^* M \otimes \mathfrak{u}(n)^\perp = \eta^\perp \otimes \mathfrak{u}(n)_{|\zeta^\perp}^\perp + \eta \otimes \mathfrak{u}(n)_{|\zeta^\perp}^\perp + \eta^\perp \otimes \eta^\perp \wedge \eta + \eta \otimes \eta^\perp \wedge \eta$$

$$T^*M \otimes \mathfrak{u}(n)^\perp = \eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp \otimes \eta^\perp \wedge \eta + \eta \otimes \eta^\perp \wedge \eta$$

$$\eta^\perp \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 \text{ (Gray\&Hervella's modules)}$$

$$\eta^\perp \otimes \eta^\perp \wedge \eta = \mathcal{C}_5 + \mathcal{C}_8 + \mathcal{C}_9 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_{10}$$

$$\eta \otimes \mathfrak{u}(n)^\perp_{|\zeta^\perp} = \mathcal{C}_{11}$$

$$\eta \otimes \eta^\perp \wedge \eta = \mathcal{C}_{12}$$

Fundamental two-form, $F = \langle \cdot, \varphi \cdot \rangle$

$$\xi^{U(n)} \rightarrow -\xi^{U(n)} F = \nabla^{U(n)} F - \xi^{U(n)} F = \nabla F,$$

D. Chinea and J. C. González-Dávila, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.* (4) 156 (1990), 15–36.

- 1 if $n = 1$, $\xi^{U(1)} \in T^*M \otimes \mathfrak{u}(1)^\perp = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{12}$;
- 2 if $n = 2$, $\xi^{U(2)} \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \cdots \oplus \mathcal{C}_{12}$;
- 3 if $n \geq 3$, $\xi^{U(n)} \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{12}$.

$$\begin{aligned}\xi_X^{U(n)} &= -\frac{1}{2}\varphi \circ \nabla_X \varphi + \nabla_X \eta \otimes \zeta - \frac{1}{2}\eta \otimes \nabla_X \zeta \\ &= \frac{1}{2}(\nabla_X \varphi) \circ \varphi + \frac{1}{2}\nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta\end{aligned}$$

if the almost contact structure is of type $\mathcal{C}_5 \oplus \cdots \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$, then

$$\xi_X^{U(n)} = \nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta$$

$$\text{Ric}^{\text{ac}}(X, Y) = \langle R_{e_i, X} \varphi e_i, \varphi Y \rangle$$

$$\text{Ric}^{\text{ac}}(\varphi X, \varphi Y) = \text{Ric}^{\text{ac}}(Y_{\zeta^\perp}, X_{\zeta^\perp}), \quad \text{Ric}^{\text{ac}}(X, \zeta) = 0$$

$$\text{Ric}^{\text{ac}}(X, Y) = \langle R_{e_i, X} \varphi e_i, \varphi Y \rangle$$

Lemma

If $(M, \langle \cdot, \cdot \rangle, \varphi, \zeta)$ is an almost contact metric $2n + 1$ -manifold, then the almost contact Ricci curvature satisfies

$$\begin{aligned} \text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) &= \langle (\nabla_{e_i}^{U(n)} \xi)_{\varphi e_i} \varphi X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle + \langle \xi_{\xi_{e_i} \varphi e_i} \varphi X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle, \\ \text{Ric}^{\text{ac}}(\zeta, X) &= \langle (\nabla_{e_i}^{U(n)} \xi)_{\varphi e_i} \eta, \varphi X \rangle + \langle \xi_{\xi_{e_i} \varphi e_i} \eta, \varphi X \rangle, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. Furthermore, we have:

- (i) The restriction $\text{Ric}_{\text{alt}|_{\zeta^\perp}}^{\text{ac}}$ of $\text{Ric}_{\text{alt}}^{\text{ac}}$ to the space ζ^\perp is in $\mathfrak{u}(n)|_{\zeta^\perp}$ and determines a $U(n)$ -component of the Weyl curvature tensor W .
- (ii) The one-form $\text{Ric}^{\text{ac}}(\zeta, \cdot)$ is in η^\perp and determines another $U(n)$ -component of W .

As a consequence, if an almost contact metric $2n + 1$ -manifold with $n > 1$ is conformally flat, i.e. $W = 0$, then $\text{Ric}_{\text{alt}|_{\zeta^\perp}}^{\text{ac}} = 0$ and $\text{Ric}^{\text{ac}}(\zeta, \cdot) = 0$, or equivalently, $\text{Ric}_{\text{alt}}^{\text{ac}} = 0$.

$$d^2F = 0, \quad (\nabla F)_{(i)} = (-\xi F)_{(i)} \rightarrow \xi_{(i)}$$

Lemma

For almost contact metric structures of type $\mathcal{C}_1 \oplus \dots \oplus \mathcal{C}_{10}$, the following identity is satisfied

$$\begin{aligned} 0 = & 3\langle (\nabla_{e_i}^{U(n)} \xi_{(1)})_{e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle - \langle (\nabla_{e_i}^{U(n)} \xi_{(3)})_{e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle \\ & + (n-2)\langle (\nabla_{e_i}^{U(n)} \xi_{(4)})_{e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle + \langle \xi_{(3)X_{\zeta^\perp}} e_i, \xi_{(1)e_i} Y_{\zeta^\perp} \rangle \\ & - \langle \xi_{(3)Y_{\zeta^\perp}} e_i, \xi_{(1)e_i} X_{\zeta^\perp} \rangle + \langle \xi_{(3)X_{\zeta^\perp}} e_i, \xi_{(2)e_i} Y_{\zeta^\perp} \rangle \\ & - \langle \xi_{(3)Y_{\zeta^\perp}} e_i, \xi_{(2)e_i} X_{\zeta^\perp} \rangle - \frac{n-5}{n-1} \langle \xi_{(1)\xi_{(4)e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle \\ & - \frac{n-2}{n-1} \langle \xi_{(2)\xi_{(4)e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle + \langle \xi_{(3)\xi_{(4)e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle \\ & + (n-2)(\xi_{(5) e_i} \eta) \wedge (\xi_{(10) e_i} \eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) - 2(\xi_{(8) e_i} \eta) \wedge (\xi_{(10) e_i} \eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) \\ & + (n-2)(\xi_{(6) e_i} \eta) \wedge (\xi_{(10) e_i} \eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}) - 2(\xi_{(7) e_i} \eta) \wedge (\xi_{(10) e_i} \eta)(X_{\zeta^\perp}, Y_{\zeta^\perp}). \end{aligned}$$

[A. Swann and FMC]

Lemma

For almost contact metric manifolds of type $\mathcal{C}_1 \oplus \dots \oplus \mathcal{C}_{10}$, the following identity is satisfied

$$\begin{aligned}
 0 = & - \left((\nabla_{e_i}^{U(n)} \xi_{(5)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) - 3 \left((\nabla_{e_i}^{U(n)} \xi_{(6)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) \\
 & - 3 \left((\nabla_{e_i}^{U(n)} \xi_{(7)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) - \left((\nabla_{e_i}^{U(n)} \xi_{(8)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) \\
 & + 3 \left((\nabla_{e_i}^{U(n)} \xi_{(9)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) + \left((\nabla_{e_i}^{U(n)} \xi_{(10)})_{e_i} \eta \right) (Y_{\zeta_{\perp}}) \\
 & - (\xi_{(6)e_i} \eta)(\xi_{(1)e_i} Y_{\zeta_{\perp}}) - (\xi_{(7)e_i} \eta)(\xi_{(1)e_i} Y_{\zeta_{\perp}}) - (\xi_{(10)e_i} \eta)(\xi_{(1)e_i} Y_{\zeta_{\perp}}) \\
 & - (\xi_{(6)e_i} \eta)(\xi_{(2)e_i} Y_{\zeta_{\perp}}) - (\xi_{(7)e_i} \eta)(\xi_{(2)e_i} Y_{\zeta_{\perp}}) + (\xi_{(10)e_i} \eta)(\xi_{(2)e_i} Y_{\zeta_{\perp}}) \\
 & + (\xi_{(5)e_i} \eta)(\xi_{(3)e_i} Y_{\zeta_{\perp}}) + (\xi_{(8)e_i} \eta)(\xi_{(3)e_i} Y_{\zeta_{\perp}}) + (\xi_{(9)e_i} \eta)(\xi_{(3)e_i} Y_{\zeta_{\perp}}) \\
 & + \frac{n}{n-1} (\xi_{(8)\xi_{(4)e_i} e_i} \eta)(Y_{\zeta_{\perp}}) - (\xi_{(10)\xi_{(4)e_i} e_i} \eta)(Y_{\zeta_{\perp}}).
 \end{aligned}$$

In particular, if the almost contact manifold is of type $\mathcal{C}_5 \oplus \cdots \oplus \mathcal{C}_{10}$, then

$$\begin{aligned} 0 &= (n-1)(\nabla_{e_i}^{U(n)} \xi_{(5)})_{e_i} + 2(\nabla_{e_i}^{U(n)} \xi_{(6)})_{e_i} + 2(\nabla_{e_i}^{U(n)} \xi_{(7)})_{e_i} \\ &\quad - (\nabla_{e_i}^{U(n)} \xi_{(8)})_{e_i} - 2(\nabla_{e_i}^{U(n)} \xi_{(9)})_{e_i} + (\nabla_{e_i}^{U(n)} \xi_{(10)})_{e_i}, \\ 0 &= (n-2)(\nabla_{e_i}^{U(n)} \xi_{(5)})_{e_i} - (\nabla_{e_i}^{U(n)} \xi_{(6)})_{e_i} - (\nabla_{e_i}^{U(n)} \xi_{(7)})_{e_i} \\ &\quad - 2(\nabla_{e_i}^{U(n)} \xi_{(8)})_{e_i} + (\nabla_{e_i}^{U(n)} \xi_{(9)})_{e_i} + 2(\nabla_{e_i}^{U(n)} \xi_{(10)})_{e_i}. \end{aligned}$$

Example: $\mathcal{C}_5 \oplus \mathcal{C}_6$ (trans-Sasakian)

$$\left. \begin{aligned} 0 &= (n-1)(\nabla_{e_i}^{U(n)} \xi_{(5)})_{e_i} + 2(\nabla_{e_i}^{U(n)} \xi_{(6)})_{e_i}, \\ 0 &= (n-2)(\nabla_{e_i}^{U(n)} \xi_{(5)})_{e_i} - (\nabla_{e_i}^{U(n)} \xi_{(6)})_{e_i}, \end{aligned} \right\} \Rightarrow \begin{aligned} \nabla_{e_i}^{U(n)} \xi_{(5)}_{e_i} &= 0, \\ \nabla_{e_i}^{U(n)} \xi_{(6)}_{e_i} &= 0 \end{aligned}$$

$$\xi_{e_i} e_i = b\zeta, \quad \xi_{\xi_{e_i} e_i} = b\xi_{\zeta} = 0$$

If $(M, \langle \cdot, \cdot \rangle, \varphi, \zeta)$ is an almost contact metric $2n + 1$ -manifold with fundamental two-form F , then the following conditions are equivalent:

- 1 The almost contact metric structure is harmonic.
- 2 $\nabla^* \nabla \varphi \in \mathfrak{u}(n) + \zeta_c^\perp$, where $\zeta_c^\perp = \{a \otimes \zeta - \eta \otimes a^\sharp \mid a \in \eta^\perp\} \cong \eta^\perp \wedge \eta$, and $\nabla^* \nabla \zeta = -\xi_{e_i} \xi_{e_i} \zeta$.
- 3 $\nabla^* \nabla F \in \mathfrak{u}(n) + \eta^\perp \wedge \eta$, i.e. $\nabla^* \nabla F(\varphi X, \varphi Y) = \nabla^* \nabla F(X_{\zeta^\perp}, Y_{\zeta^\perp})$, and $\nabla^* \nabla \eta = -\xi_{e_i}(\xi_{e_i} \eta)$.
- 4 For all $X, Y \in \mathfrak{X}(M)$, we have:

$$\langle (\nabla_{e_i}^{U(n)} \xi)_{e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle + \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle = 0,$$

$$(\nabla_{e_i}^{U(n)} \xi)_{e_i} \eta + \xi_{\xi_{e_i} e_i} \eta = 0.$$

In particular, if the structure is of type $\mathcal{C}_5 \oplus \dots \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$, then it is harmonic if and only if $\nabla^* \nabla \zeta = \|\nabla \zeta\|^2 \zeta$, that is, ζ is a harmonic unit vector field.

E. Vergara-Díaz and C. M. Wood, Harmonic Almost Contact Structures, *Geom. Dedicata* 123 (2006), 131–151.

Theorem

For an almost contact metric $2n + 1$ -manifold $(M, \langle \cdot, \cdot \rangle, \varphi, \zeta)$, we have:

- (i) If M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact structure is harmonic if and only if $\text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 0$ and $\text{Ric}^{\text{ac}}(\zeta, X) = 0$, for all $X, Y \in \mathfrak{X}(M)$.
- (ii) For $n \neq 2$, if M is of type $\mathcal{C}_1 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, then the almost contact metric structure is harmonic if and only if

$$(n-1)(n-5) \text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2(n+1)(n-3) \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle,$$

$$\text{Ric}^{\text{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i} \eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

- (iii) For $n \neq 2$, if M is of type $\mathcal{C}_1 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then almost contact metric structure is harmonic if and only if

$$(n-1)(n-5) \text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2(n+1)(n-3) \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle,$$

$$\text{Ric}^{\text{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i} \eta)(X_{\zeta^\perp}),$$

for all $X, Y \in \mathfrak{X}(M)$.

- (iv) For $n \neq 2$, if M is of type $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, then the almost contact metric structure is harmonic if and only if

$$(n-1) \text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = 2n \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle,$$

$$\text{Ric}^{\text{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i} \eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

- (v) For $n \neq 2$, if M is of type $\mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is harmonic if and only if, for all $X, Y \in \mathfrak{X}(M)$,

$$\frac{n-1}{2n} \text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = \langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle, \quad \text{Ric}^{\text{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i} \eta)(X).$$

(vi) If M is normal ($\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$), then the almost contact metric structure is harmonic if and only if

$$\text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = -2\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle, \quad \text{Ric}^{\text{ac}}(\zeta, X) = -2(\xi_{\xi_{e_i} e_i} \eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(vii) If M is of type $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is harmonic if and only if

$$\text{Ric}_{\text{alt}}^{\text{ac}}(X_{\zeta^\perp}, Y_{\zeta^\perp}) = -2\langle \xi_{\xi_{e_i} e_i} X_{\zeta^\perp}, Y_{\zeta^\perp} \rangle, \quad \text{Ric}^{\text{ac}}(\zeta, X) = 2(\xi_{\xi_{e_i} e_i} \eta)(X),$$

for all $X, Y \in \mathfrak{X}(M)$.

(viii) If M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9, \mathcal{C}_1 \oplus \mathcal{C}_6 \oplus \mathcal{C}_8$, then the almost contact structure is harmonic if and only if $\text{Ric}^{\text{ac}}(\zeta, X) = 0$, for all $X \in \mathfrak{X}(M)$.

(ix) For $n \neq 2$, if M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6, \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7, \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9, \mathcal{C}_4 \oplus \mathcal{C}_8$, then the almost contact structure is harmonic if and only if $\text{Ric}^{\text{ac}}(\zeta, X) = 0$, for all $X \in \mathfrak{X}(M)$.

In particular:

- (i)* If M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_5, \mathcal{C}_1 \oplus \mathcal{C}_8, \mathcal{C}_1 \oplus \mathcal{C}_9, \mathcal{C}_3 \oplus \mathcal{C}_6, \mathcal{C}_3 \oplus \mathcal{C}_7, \mathcal{C}_3 \oplus \mathcal{C}_{10}, \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7, \mathcal{C}_5 \oplus \mathcal{C}_8, \mathcal{C}_5 \oplus \mathcal{C}_9, \mathcal{C}_5 \oplus \mathcal{C}_{10}, \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8, \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10}, \mathcal{C}_8 \oplus \mathcal{C}_9, \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact structure is harmonic.
- (ii)* For $n \neq 2$, if M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5, \mathcal{C}_4 \oplus \mathcal{C}_6, \mathcal{C}_4 \oplus \mathcal{C}_7, \mathcal{C}_4 \oplus \mathcal{C}_9$, then the almost contact structure is harmonic.

Corollary

- (a) *If an almost contact metric structure is of type \mathcal{D} , for $\mathcal{D} = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7$, $\mathcal{C}_5 \oplus \mathcal{C}_8$, $\mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_5 \oplus \mathcal{C}_{10}$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10}$, $\mathcal{C}_8 \oplus \mathcal{C}_9$, $\mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the characteristic vector field ζ is a harmonic unit vector field.*
- (b) *For a locally conformally flat $2n + 1$ -manifold $(M, \langle \cdot, \cdot \rangle)$ with $n > 1$:*
- If an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, $\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, then it is harmonic.*
 - For $n > 2$, if an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7$, $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$, $\mathcal{C}_4 \oplus \mathcal{C}_8$, then it is harmonic.*

$$2d^* \text{Ric} + ds = 0$$

$$\text{weakly-ac-Einstein, } \text{Ric}^{\text{ac}}(X, Y) = \frac{1}{2n} s^{\text{ac}}(\langle X, Y \rangle - \eta(X)\eta(Y))$$

Lemma

For almost contact manifolds of type $\mathcal{C}_1 \oplus \dots \oplus \mathcal{C}_{10}$, we have

$$2d^*(\text{Ric}^{\text{ac}})^t(X) + ds^{\text{ac}}(X) = 2\langle R_{(e_i, X)}, \xi_{\varphi e_i} \varphi \rangle - 4 \text{Ric}^{\text{ac}}(X, \xi_{e_i} e_i) \\ + 4\langle \text{Ric}^{\text{ac}}, \xi_X \rangle - 2d^* F(\zeta) \text{Ric}^{\text{ac}}(\zeta, \varphi X).$$

where $(\text{Ric}^{\text{ac}})^t(X, Y) = \text{Ric}^{\text{ac}}(Y, X)$ and $\xi_X^b(Y, Z) = \langle \xi_X Y, Z \rangle$. In particular, if the manifold is weakly-ac-Einstein, then

$$(n-1)ds^{\text{ac}}(X) + (ds^{\text{ac}}(\zeta) + s^{\text{ac}} d^* \eta)\eta(X) = 2n\langle R_{(e_i, X)}, \xi_{\varphi e_i} \varphi \rangle - 2s^{\text{ac}} \langle \xi_{e_i} e_i, X \rangle.$$

For an almost contact metric $2n + 1$ -manifold $(M, \langle \cdot, \cdot \rangle, \varphi, \zeta)$, we have:

- (i) If M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ or $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is a harmonic map if and only if it is a harmonic almost contact structure and $2d^* \text{Ric}^{\text{ac}} + ds^{\text{ac}} = 0$. In particular:
 - (a) If the structure is of type $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ and the manifold is weakly- ac -Einstein, then the almost contact structure metric structure is a harmonic map if and only if s^{ac} is such that $n ds^{\text{ac}} = s^{\text{ac}} d^* \eta \eta = -s^{\text{ac}} \zeta_{e_i}^\sharp e_i$.
 - (b) If the structure is of type $\mathcal{D} = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ and the manifold is weakly- ac -Einstein, then the almost contact metric structure is a harmonic map if and only if s^{ac} is constant.
 - (c) If the manifold is nearly-K-cosymplectic (\mathcal{C}_1), then almost contact metric structure is a harmonic map. Furthermore, if the nearly-K-cosymplectic structure is flat, then it is cosymplectic, i.e., $\xi = 0$.

- (ii) If M is of type \mathcal{D} , where $\mathcal{D} = \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ or $\mathcal{D} = \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$, then the almost contact metric structure is a harmonic map if and only if it is a harmonic structure and

$$2d^*(\text{Ric}^{\text{ac}})^t(X) + ds^{\text{ac}}(X) + 4\text{Ric}^{\text{ac}}(X, \xi_{e_j} e_j) - 4\langle \text{Ric}^{\text{ac}}, \xi_X^b \rangle + 2d^*F(\zeta)\text{Ric}^{\text{ac}}(\zeta, \varphi X) = 0,$$

for all $X \in \mathfrak{X}(M)$. In particular,

- (a)* If Ric^{ac} is symmetric, then the almost contact structure is a harmonic map if and only if $\xi_{\xi_{e_j} e_i} = 0$ and $2d^*\text{Ric}^{\text{ac}} + ds^{\text{ac}} + 4\xi_{e_j} e_i \lrcorner \text{Ric}^{\text{ac}} = 0$. Furthermore, if the manifold is weakly- ac -Einstein, then the almost contact structure is a harmonic map if and only if $\xi_{\xi_{e_j} e_i} = 0$ and $(n-1)ds^{\text{ac}} + (ds^{\text{ac}}(\zeta) - s^{\text{ac}}d^*\eta)\eta + 2s^{\text{ac}}\xi_{e_i}^b e_j = 0$.
- (b)* If the almost contact metric structure is of type $\mathcal{C}_3 \oplus \mathcal{C}_i$, $i = 6, 7, 10$, then it is a harmonic map if and only if

$$2d^*\text{Ric}^{\text{ac}} + ds^{\text{ac}} = 0.$$

For an oriented and compact Riemannian n -manifold $(M, \langle \cdot, \cdot \rangle)$ equipped with a G -structure, $G \subseteq SO(n)$ and a differential p -form ϕ preserved by the action of G , the following Bochner type formula is satisfied

$$\int_M \left(\frac{1}{p+1} \|d\phi\|^2 + p \|d^*\phi\|^2 - \|\nabla\phi\|^2 \right) = \int_M \langle \tilde{\mathcal{R}}\phi, \phi \rangle$$

G. Bor and L. Hernández Lamonedá, Bochner formulae for orthogonal G -structures on compact manifolds, *Diff. Geom. Appl.* **15** (2001), 265–286.

$$\text{alt } \alpha(x_1, \dots, x_p) = \sum_{\tau} \text{sign}(\tau) \alpha(x_{\tau(1)}, \dots, x_{\tau(p)})$$

$$(R^c \phi)(x_1, \dots, x_p) = (R(x_1, e_i)\phi)(e_i, x_2, \dots, x_p)$$

$$\tilde{\mathcal{R}}(\phi) = \text{alt}(R^c \phi)$$

$$\int_M \left(\frac{1}{3} \|dF\|^2 + 2 \|d^*F\|^2 - \|\nabla F\|^2 \right) = 2 \int_M (s - s^{\text{ac}} - \text{Ric}(\zeta, \zeta)).$$

$$\int_M \left(\frac{1}{2} \|d\eta\|^2 + \|d^*\eta\|^2 - \|\nabla \eta\|^2 \right) = \int_M \text{Ric}(\zeta, \zeta).$$

$$\begin{aligned} & \int_M \left(8 \|\xi_{(1)}\|^2 - 4 \|\xi_{(2)}\|^2 + \|\xi_{(5)}\|^2 + (2n-1) \|\xi_{(6)}\|^2 - \|\xi_{(7)}\|^2 + \|\xi_{(8)}\|^2 \right. \\ & \left. - \|\xi_{(9)}\|^2 - \|\xi_{(10)}\|^2 - 4 \|\xi_{(11)}\|^2 - \|\xi_{(12)}\|^2 + \|2(\nabla \eta)_{(10)} \circ \varphi - (\nabla_\zeta F)_{(11)}\|^2 \right. \\ & \left. + 2 \| -e_i \lrcorner (\nabla_{e_i} F)_{(4)} + (\nabla_\zeta \eta)_{(12)} \circ \varphi \|^2 \right) = 2 \int_M (s - s^{\text{ac}} - \text{Ric}(\zeta, \zeta)) \end{aligned}$$

$$\begin{aligned} & \int_M \left((2n-1) \|\xi_{(5)}\|^2 + \|\xi_{(6)}\|^2 + \|\xi_{(7)}\|^2 - \|\xi_{(8)}\|^2 - \|\xi_{(9)}\|^2 + \|\xi_{(10)}\|^2 \right) \\ & = 2 \int_M \text{Ric}(\zeta, \zeta) \end{aligned}$$

Theorem

Let $(M, \langle \cdot, \cdot \rangle)$ be a $(2n + 1)$ -dimensional conformally flat compact Riemannian manifold, where $n > 1$. If s is the scalar curvature and $C_{\langle \cdot, \cdot \rangle} = \int_M s$, that is, a constant depending on the metric $\langle \cdot, \cdot \rangle$, then every almost contact structure σ compatible with $\langle \cdot, \cdot \rangle$ satisfies

$$\begin{aligned} \frac{2(n-1)}{2n-1} C_{\langle \cdot, \cdot \rangle} &= \int_M \left(4\|\xi_{(1)}(\sigma)\|^2 - 2\|\xi_{(2)}(\sigma)\|^2 + (n-1)\|\xi_{(5)}(\sigma)\|^2 \right. \\ &+ \left(n - \frac{1}{2n-1} \right) \|\xi_{(6)}(\sigma)\|^2 - \frac{1}{2n-1} \|\xi_{(7)}(\sigma)\|^2 + \frac{1}{2n-1} \|\xi_{(8)}(\sigma)\|^2 \\ &- \left(1 - \frac{1}{2n-1} \right) \|\xi_{(9)}(\sigma)\|^2 - \frac{1}{2n-1} \|\xi_{(10)}(\sigma)\|^2 - 2\|\xi_{(11)}(\sigma)\|^2 \\ &- \|\xi_{(12)}(\sigma)\|^2 + \frac{1}{2} \|2(\nabla \eta_\sigma)_{(10)} \circ \varphi_\sigma - (\nabla_{\zeta_\sigma} F_\sigma)_{(11)}\|^2 \\ &\left. + \| -e_i \lrcorner (\nabla_{e_i} F_\sigma)_{(4)} + (\nabla_{\zeta_\sigma} \eta_\sigma)_{(12)} \circ \varphi_\sigma \|^2 \right). \end{aligned}$$

Theorem

Moreover:

- (i) If σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_1 \oplus \mathcal{C}_4$ and $n = 3$, then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{1}{10} C_{\langle \cdot, \cdot \rangle}$. Furthermore, in this situation any other energy minimiser is of type $\mathcal{C}_1 \oplus \mathcal{C}_4$.
- (ii) If $n = 2$ or $n \geq 4$, and σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type \mathcal{C}_4 , then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{1}{2(2n-1)} C_{\langle \cdot, \cdot \rangle}$. Furthermore, in this situation any other energy minimiser is of type \mathcal{C}_4 .
- (iii) If σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type \mathcal{C}_2 , then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = -\frac{n-1}{2(2n-1)} C_{\langle \cdot, \cdot \rangle}$. Furthermore, in this situation any other energy minimiser is of type \mathcal{C}_2 .

Theorem

Let $(M, \langle \cdot, \cdot \rangle)$ be a $(2n + 1)$ -dimensional compact Einstein manifold. If s is the scalar curvature, then every almost contact structure σ compatible with $\langle \cdot, \cdot \rangle$ satisfies

$$\int_M \left((2n - 1) \|\xi_{(5)}(\sigma)\|^2 + \|\xi_{(6)}(\sigma)\|^2 + \|\xi_{(7)}(\sigma)\|^2 - \|\xi_{(8)}(\sigma)\|^2 - \|\xi_{(9)}(\sigma)\|^2 + \|\xi_{(10)}(\sigma)\|^2 \right) = \frac{2s}{2n+1} \text{Vol}(M).$$

Moreover:

- (i) For $n = 1$, if σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_5 \oplus \mathcal{C}_6$ (trans-Sasakian), then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{s}{3} \text{Vol}(M)$. Furthermore, in this situation any other energy minimiser is trans-Sasakian.
- (ii) For $n > 1$, if σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type \mathcal{C}_5 (α -Kenmotsu, where $2n\alpha = -d^*\eta_{\sigma_0}$), then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = \frac{s}{4n^2-1} \text{Vol}(M)$. Furthermore, in this situation any other energy minimiser is of type α -Kenmotsu with $4n \int_M \alpha^2 = \frac{s}{4n^2-1} \text{Vol}(M)$.
- (iii) If σ_0 is an almost contact structure compatible with $\langle \cdot, \cdot \rangle$ of type $\mathcal{C}_8 \oplus \mathcal{C}_9$, then σ_0 is an energy minimiser such that its total bending is $B(\sigma_0) = -\frac{s}{2n+1} \text{Vol}(M)$. Furthermore, in this situation any other energy minimiser is of type $\mathcal{C}_8 \oplus \mathcal{C}_9$.

Example

- ① \mathbb{C}^{n+1} , $(\langle \cdot, \cdot \rangle, J)$ $S^{2n+1}(r)$, U is a unit normal vector field, on $S^{2n+1}(r)$, α -Sasakian structure, $\langle \cdot, \cdot \rangle$, $\varphi = \tan \circ J$, $\zeta = JU$

$$a^2 = \frac{1}{r^2}, \quad \text{Ric} = \frac{2n}{r^2} \langle \cdot, \cdot \rangle, \quad \text{Ric}^{\text{ac}} = \frac{1}{r^2} (\langle \cdot, \cdot \rangle - \eta \otimes \eta).$$

$S^{2n+1}(r)$ is Einstein, ac-Einstein and conformally flat

$\pm \frac{1}{r}$ -Sasakian structure is a harmonic map

- ② Different generalisations of the Heisenberg group admit almost contact structures of type $\mathcal{C}_6 \oplus \mathcal{C}_7$ and $\mathcal{C}_8 + \mathcal{C}_9$. Therefore, they are harmonic almost contact structures. Most of them satisfy $2d^* \text{Ric}^{\text{ac}} + ds^{\text{ac}} = 0$. As a consequence, these almost contact structures are harmonic maps. The characteristic vector fields are harmonic unit vector fields.
- ③ $S^6 \times S^1$ admits a structure of strict type \mathcal{C}_1 . Therefore, it is an energy minimiser.
- ④ $S^{2n+1} \times S^1 \times S^1$ admits a structure of strict type \mathcal{C}_4 . Therefore, it is an energy minimiser.