Homogeneous para-Kähler Einstein manifolds

Dmitri V. Alekseevsky

Hamburg,14-18 July 2008

The talk is based on a joint work with C.Medori and A.Tomassini (Parma)

See ArXiv 0806.2272, where also a survey of recent results on para-complex geometry is given. It includes

para-complex and generalized para-complex structures,

para-hypercomplex (complex product structure) and 3-webs,

almost para-Hermitian and para-Hermitian structures,

para-Kähler (bi-Lagrangian) structures,

para hyperKähler and para hyperKahler with torsion structures,

para-quaternionic Kahler and para-quaterniuonic Kähler with torsion structures,

para-CR and para-quaternionbic CR structures.

Para-complex structures on a real vector space

The algebra of para-complex numbers is defined by

$$C = \mathbb{R} + e\mathbb{R} \simeq \mathbb{R}^2,\ e^2 = 1.$$

Para-complex structure in a vector space V :

$$K : V \to V, \quad K^2 = 1$$

such that

$$V = V^+ + V^-, \quad \dim V^+ = \dim V^-.$$

Para-complexification of (V, K) is

$$V^C := V \otimes C.$$

Holomorphic and antiholomorphic subspaces are defined by

$$V^{1,0} = \{v \in V^C \mid Kv = ev\} = \{v + eKv \mid v \in V\},$$
$$V^{0,1} = \{v \in V^C \mid Kv = -ev\} = \{v - eKv \mid v \in V\},$$

Then $V^C = V^{1,0} \oplus V^{0,1}$.

Exterior forms

The dual space

$$(V^*)^C = V^* \otimes C = V_{1,0} \oplus V_{0,1},$$

where

$$V_{1,0} = \{\alpha \in V^{*C} \,|, K^*\alpha = e\alpha\} = \{\alpha + eK^*\alpha \mid \alpha \in V^*\},$$
$$V_{0,1} = \{\alpha \in V^{*C} \,|, K^*\alpha = -e\alpha\} = \{\alpha - eK^*\alpha \mid \alpha \in V^*\}.$$

Denote by $\wedge^{p,q}V^{*C}$ the subspace of $\wedge V^{*C}$ spanned by $\alpha \wedge \beta$, with $\alpha \in \wedge^p V_{1,0}$ and $\beta \in \wedge^q V_{0,1}$. Then

$$\wedge^r V^{*C} = \bigoplus_{p+q=r} \wedge^{p,q} V^{*C}\,.$$

Para-Hermitian forms

Definition 1 *A* para-Hermitian form *on V^C is a map $h : V^C \times V^C \to C$ such that:*

i) h is C-linear in the first entry and C-antilinear in the second entry;

ii) $h(W, Z) = \overline{h(Z, W)}$.

Definition 2 *A* para-Hermitian symmetric form *on V^C is a symmetric C-bilinear form $h : V^C \times V^C \to C$ such that*

$$\begin{aligned} h(V^{1,0}, V^{1,0}) &= h(V^{0,1}, V^{0,1}) = 0\,, \\ h(\overline{Z}, \overline{W}) &= \overline{h(Z, W)} \end{aligned}$$

for any $Z, W \in V^C$.
It is called non-degenerate *if it has trivial kernel.*

If $h(Z, W)$ is a para-Hermitian symmetric form, then $\hat{h}(Z, W) = h(Z, \overline{W})$ is a para-Hermitian form.

Lemma 3 *There exists a natural* $1-1$ *correspondence between pseudo-Euclidean metric* g *on a vector space* V *such that*

$$g(KX, KY) = -g(X, Y)\,, \quad X, Y \in V$$

and non-degenerate para-Hermitian symmetric forms $h = g^C$ *in* V^C*, where* g^C *is the natural extension of* g *to* C*-bilinear symmetric form. Moreover, the natural* C*-extension* ω^C *of the two form* $\omega = g \circ K$ *coincides with the* $(1,1)$*-form* $g^C \circ K$*.*

Para-complex manifolds

Definition 4 *A para-complex structure on a* $2n$*-dimensional manifold* M *is a field* K *of para-complex structures such that the* ± 1*-eigen-distributions* $T^{\pm}M$ *are involutive.*
A map $f : (M, K) \to (M', K')$ *between two para-complex manifolds is said to be para-holomorphic if*

$$df \circ K = K' \circ df \,. \tag{1}$$

By Frobenius theorem, there are local coordinates (called adapted coordinates) $(z_{+}^{\alpha}, z_{-}^{\alpha})$ $\alpha = 1, \ldots, n$ on M, such that

$$T^{+}M = \mathrm{span}\left\{\frac{\partial}{\partial z_{+}^{\alpha}}, \alpha = 1, \ldots, n\right\}$$

$$T^{-}M = \mathrm{span}\left\{\frac{\partial}{\partial z_{-}^{\alpha}}, \alpha = 1, \ldots, n\right\}.$$

$(p+, q-)$-decomposition of real differential forms) The cotangent bundle T^*M splits as $T^*M = T^*_+M \oplus T^*_-M$, and , moreover,

$$\wedge^r T^*M = \bigoplus_{p+q=1}^{r} \wedge^{p,q}_{+\,-} T^*M\,,$$

where $\wedge^{p,q}_{+\,-} T^*M = \wedge^p(T^*_+M) \otimes \wedge^q(T^*_-M)$. We put

$$\begin{aligned} \partial_+ &= \mathrm{pr}_{\wedge^{p+1,q}_{+\,-}(M)} \circ d : \wedge^{p,q}_{+\,-}(M) \to \wedge^{p+1,q}_{+\,-}(M) \\ \partial_- &= \mathrm{pr}_{\wedge^{p,q+1}_{+\,-}(M)} \circ d : \wedge^{p,q}_{+\,-}(M) \to \wedge^{p,q+1}_{+\,-}(M)\,. \end{aligned}$$

Then exterior differential d can be decomposed as $d = \partial_+ + \partial_-$ and, since $d^2 = 0$, we have

$$\partial_+^2 = \partial_-^2 = 0\,, \quad \partial_+\partial_- + \partial_-\partial_+ = 0\,.$$

Para-holomorphic coordinates

Let (z_+^α, z_-^α) be adapted local coordinates on (M, K). Then

$$z^\alpha = \frac{z_+^\alpha + z_-^\alpha}{2} + e\,\frac{z_+^\alpha - z_-^\alpha}{2}\,, \quad \alpha = 1, \ldots, n\,. \tag{2}$$

z^α are para-holomorphic functions in the sense of (1) and the transition functions between two para-holomorphic coordinate systems are para-holomorphic. The real part x^α and the imaginary part y^α of the functions z^α, given by

$$x^\alpha = \frac{1}{2}(z^\alpha + \overline{z}^\alpha) = \frac{1}{2}(z_+^\alpha + z_-^\alpha),$$

$$y^\alpha = \frac{1}{2e}(z^\alpha - \overline{z}^\alpha) = \frac{1}{2}(z_+^\alpha - z_-^\alpha),$$

are not necessarily real analytic.

Para-complex differential forms

The para-complex tangent bundle

$T^C M = TM \otimes C$ is decomposed into a direct sum

$$T_p^C M = T_p^{1,0} M \oplus T_p^{0,1} M\,, \qquad (3)$$

of para-holomorphic and para-anti-holomorphic bundles, where

$$T_p^{1,0} M = \{Z \in T_p^C M \mid KZ = eZ\} =$$

$$\{X + eKX \mid X \in T_p M\}$$

$$T_p^{0,1} M = \{Z \in T_p^C M \mid KZ = -eZ\} =$$

$$\{X - eKX \mid X \in T_p M\}$$

are the "eigenspaces" of K with "eigenvalues" $\pm e$.

Similar

$$(T^C)^*M = \wedge^{1,0}(M) \oplus \wedge^{0,1}(M)\,,$$

where 1-forms

$$dz^\alpha = dx^\alpha + e dy^\alpha \quad \text{and} \quad d\overline{z}^\alpha = dx^\alpha - e dy^\alpha$$

form a basis of $\wedge^{1,0}(M)$ and $\wedge^{0,1}(M)$ dual to the bases $\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial \overline{z}^\alpha}$ respectively.

The last decomposition induces a splitting of the bundle $\wedge^r(T^C)^*M$ of para-complex r-forms on (M,K) given by

$$\wedge^r(T^C)^*M = \bigoplus_{p+q=r} \wedge^{p,q}(M)\,.$$

The sections of $\wedge^{p,q}(M)$ are called (p,q)-*forms* on the para-complex manifold (M,K). We have

$$\wedge^{1,1}_{+\,-}(M) = \{\omega \in \wedge^{1,1}(M) \mid \omega = \overline{\omega}\}\,. \qquad (4)$$

The exterior derivative $d : \wedge^r T^* M^C \to \wedge^{r+1} T^* M^C$ splits as $d = \partial + \overline{\partial}$, where

$$\begin{aligned} \partial &= \mathrm{pr}_{\wedge^{p+1,q}(M)} \circ d : \wedge^{p,q}(M) \to \wedge^{p+1,q}(M)\,, \\ \overline{\partial} &= \mathrm{pr}_{\wedge^{p,q+1}(M)} \circ d : \wedge^{p,q}(M) \to \wedge^{p,q+1}(M)\,, \end{aligned}$$

and

$$\partial^2 = 0\,, \quad \overline{\partial}^2 = 0\,, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0\,.$$

The operators $\partial, \overline{\partial}$ are related to ∂_+, ∂_- by

$$\partial = \frac{1}{2}((\partial_+ + \partial_-) + e(\partial_+ - \partial_-))$$

$$\overline{\partial} = \frac{1}{2}((\partial_+ + \partial_-) - e(\partial_+ - \partial_-))\,.$$

In particular,

$$\partial\overline{\partial} = e\,\partial_+\partial_-\,.$$

Dolbeault lemma

(V.Cortes, C. Mayer, Th. Mohaupt, F.Sauressig)

Lemma 5 *Let* (M, K) *be a para-complex manifold and* ω *be a closed 2-form belonging to* $\wedge^{1,1}_{+-}(M)$. *Then locally there exists a real-valued function* F *(called* potential*) such that*

$$\omega = \partial_+ \partial_- F = e\, \partial \overline{\partial} F\,.$$

The potential F *is defined up to addition of a function* f *satisfying the condition* $\partial_+ \partial_- f = 0$.

Para-Kähler manifolds

Definition 6 *A* para-Kähler manifold *is given equivalently by:*

i) a pseudo-Riemannian manifold (M, g) *together with a skew-symmetric para-complex structure* K *which is parallel with respect to the Levi-Civita connection;*

ii) a symplectic manifold (M, ω) *together with two complementary involutive Lagrangian distributions* $L^{\pm}$*.*

iii) a para-complex manifold (M, K) *together with a symplectic form* ω *which belongs to* $\wedge^{1,1}_{+-}(M)$*;*

Curvature and Ricci curvature of a para-Kähler manifold

Proposition 7 *The curvature R and the Ricci tensor S of a para-Kähler metric g satisfy the following*

$$R(X,Y)\circ K = K\circ R(X,Y) \quad R(KX,KY) = -R(X,Y)$$

$$S(KX,KY) = -S(X,Y)$$

for any vector fields $X,Y \in \mathfrak{X}(M)$.

We define the *Ricci form* ρ of the para-Kähler metric g by

$$\rho := \mathrm{R}ic \circ K\,. \tag{5}$$

It is a 2-form. Its para-complex extension ρ has type $(1,1)$ and in local para-holomorphic coordinates is given by

$$\rho = 2e\,\mathrm{R}ic_{\alpha\overline{\beta}} dz^{\alpha} \wedge d\overline{z}^{\beta}\,.$$

Proposition 8 *The Ricci form of a para-Kähler manifold is a closed* $(1,1)$*-form and can be represented by*

$$\rho = e\,\partial\overline{\partial}\,\mathrm{log}(\mathrm{det}(g_{\alpha\overline{\beta}}))\,. \tag{6}$$

In particular,

$$\mathrm{R}ic_{\,\alpha\overline{\beta}} = -\frac{\partial^2\,\mathrm{log}(\mathrm{det}(g_{\alpha\overline{\beta}}))}{\partial z^\alpha \partial \overline{z}^\beta}\,. \tag{7}$$

The canonical form of a para-complex manifold with a volume form

Let (M, K, vol) be an oriented manifold with para-complex structure K and a (real) volume form vol.
Let $z = (z^1, \ldots, z^n)$ be local para-holomorphic coordinates and (x^α, y^α) corresponding real coordinates, where $z^\alpha = x^\alpha + e y^\alpha$.
Then

$$\begin{aligned} \mathrm{vol} &= V(z, \overline{z}) dz^1 \wedge d\overline{z}^1 \wedge \ldots \wedge dz^n \wedge d\overline{z}^n \quad (8) \\ &= U(x, y) dx^1 \wedge dy^1 \wedge \ldots \wedge dx^n \wedge dy^n \,. \end{aligned}$$

We may assume that $U(x, y) > 0$, as M is oriented.
Then

$$V(z, \overline{z}) = (-2e)^n U(x, y) \,.$$

In particular, the function $\tilde{V} = (-e)^n V$ is positive.

Definition of the canonical 2-form

Lemma 9 *The formula*

$$\rho = e\,\partial\overline{\partial}\,\mathrm{log}\,((-e)^n V) \qquad (9)$$

defines a real global closed 2*-form of type* $(1,1)$ *on the oriented para-complex manifold* (M, K, vol)*.*

The form ρ is called the canonical form of (M, K, vol).

Corollary 10 *Let* (M, K, ω, g) *be an oriented para-Kähler manifold and denote by* vol^g *the volume form associated with the metric* g*. Then the Ricci form* ρ *of the para-Kähler manifold* M *coincides with the canonical form associated with the volume form* vol^g*. In particular* ρ *depends only on the para-complex structure and the volume form.*

A formula for the canonical form ρ in term of divergence

Lemma 11 *Let X, Y be real vector fields with* $\operatorname{div} X = \operatorname{div} Y = 0$. *Assume that the fields* $X^c = X + eKX\,, Y^c = Y + eKY$ *are para-holomorphic. Then*

$$2\rho(X,Y) = \operatorname{div}\left(K[X,Y]\right). \tag{10}$$

Koszul formula for the canonical form ρ of a homogeneous para-complex manifold $(M = G/H, K)$ with an invariant volume form vol.

Given $(M = G/H, K, \mathrm{vol})$. Let $\mathfrak{m}$ be a complementary subspace to the Lie subalgebra $\mathfrak{h} = LieH$ in the Lie algebra $\mathfrak{g} = LieG$. We identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$, $o = eH$ and extend $K_o \in \mathrm{E}nd(\mathfrak{m})$ to an endomorphism $\tilde{K}$ of $\mathfrak{g}$ with kernel $\mathfrak{h}$.

Then the pull back ρ of the canonical 2-form associated with (vol, K) at the point $e \in G$ is given by

$$2\rho_e(X,Y) = \sum \omega^i \left([\tilde{K}[X,Y], X_i] - \tilde{K}[[X,Y],X_i] \right).$$

In particular,

$$\rho_e = d\psi\,,$$

where $\psi \in \mathfrak{g}^*$ is the $\mathrm{Ad}_{\mathfrak{h}}$-invariant 1-form on $\mathfrak{g}$ given by

$$\psi(X) = -\,\mathrm{tr}_{\,\mathfrak{g}/\mathfrak{h}}\left(\mathrm{ad}_{\tilde{K}X} - \tilde{K}\mathrm{ad}_X\right)\,, \quad \forall\, X \in \mathfrak{g}\,. \tag{11}$$

The 1-form $\psi \in \mathfrak{g}^*$ is called the Koszul form .

A description of homogeneous para-Kähler manifolds $(M = G/H, \omega, L^{\pm})$ of a semisimple Lie group

We recall an important characterization of homogeneous manifolds $M = G/H$ of a semisimple Lie group G which admit invariant para-Kähler structure $(\omega, L^{\pm})$.

Theorem 12 (Hou-Deng-Kaneyuki-Nishiyama,97) *A homogeneous manifold $M = G/H$ of a semisimple group G admits an invariant para-Kähler structure $(\omega, L^{\pm})$ iff it is a covering of the adjoint orbit $\mathrm{Ad}_G h = G/Z_G(h)$ of a semisimple element $h \in \mathfrak{g}$.*

The proof follows from a result by Ozeki and Wakimoto (1972) that any polarization of a semisimple Lie algebra is a parabolic subalgebra and a classical result by Dixmier that the intersection of two parabolic subalgebras has maximal rank. Indeed,the subalgebras

$$\mathfrak{p}^{\pm} := \{X \in \mathfrak{g} \,|\, X^*_o \in L^{\pm}|_o\}$$

are polarizations for 1-form $\xi \in \mathfrak{g}^*$ such that $\omega_o = d\xi$, that is maximal isotropic subspaces with respect to $\omega_o = d\xi \in \Lambda^2\mathfrak{g}^*$.

Fundamental gradations of a semisimple Lie algebra

A $\mathbb{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} + \cdots + \mathfrak{g}^{k} \quad [\mathfrak{g}^{i}, \mathfrak{g}^{j}] \subset \mathfrak{g}^{i+j} \tag{12}$$

of a (real or complex) semi-simple Lie algebra $\mathfrak{g}$ is called
fundamental if the subalgebra

$$\mathfrak{g}^{-} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^{-1}$$

is generated by $\mathfrak{g}^{-1}$.

There exist unique element $d \in \mathfrak{g}$ (called the grading element) such that

$$\mathrm{Ad}_d \big|_{\mathfrak{g}^j} = j \, \mathrm{Id}\,.$$

We set

$$\mathfrak{g}^{\pm} = \sum_{\pm j > 0} \mathfrak{g}^{j}.$$

Then $\mathfrak{g} = \mathfrak{g}^{-} + \mathfrak{g}^{0} + \mathfrak{g}^{+}$ (associated decomposition)

Examples. Fundamental gradations of $\mathfrak{sl}(V)$

Let V be a (complex or real) vector space and $V = V^1 + \cdots + V^k$ a decomposition of V into a direct sum of subspaces. It defines a fundamental gradation $\mathfrak{sl}(V) = \sum_{i=-k}^{k} \mathfrak{g}^i$ of the Lie algebra $\mathfrak{sl}(V)$, where

$$\mathfrak{g}^i = \{A \in \mathfrak{sl}(V),\ AV^j \subset V^{i+j},\ j = 1, \ldots, k\ \}\ .$$

A generalized Gauss decomposition A direct space decomposition

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$$

is called a generalized Gauss decomposition if $\mathfrak{p}^\pm := \mathfrak{n}^\pm + \mathfrak{h}$ are parabolic subalgebras with nilradical $\mathfrak{n}^\pm$ and reductive part $\mathfrak{h}$.

Proposition 13 *Any generalized Gauss decomposition (gGd) is associated with a unique fundamental gradation i.e.* $\mathfrak{n}^\pm = \mathfrak{g}^\pm$, $\mathfrak{h} = \mathfrak{g}^0$.

Invariant para-Kähler structures and generalized Gauss decompositions

Proposition 14 *Let $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ be a generalized Gauss decomposition associated with a fundamental gradation, defined by a grading element $d \in \mathfrak{g}$. Let G be a Lie group with the Lie algebra $\mathfrak{g}$ and H a closed subgroup of G with* $\mathrm{Lie}H = \mathfrak{h}$ *which preserves $d \in \mathfrak{g}$. Then $M = G/H$ has an invariant para-complex structure K defined by $K|_{\mathfrak{n}^\pm} = \pm \mathrm{Id}$. Moreover, any* Ad_H*-invariant element $h \in \mathfrak{h}$ with $Z_{\mathfrak{g}}(h) = \mathfrak{h}$ defines an invariant symplectic form ω^h on $M = G/H$ (where $\omega^h_o = dB \circ h$ and B is the Killing form) which is consistent with K , i;.e. (K, ω) is an invariant para-Kähler structure.*
Moreover, any invariant para-Kähler structure cab be obtained by this construction.

Fundamental gradations of a complex semisimple Lie algebra $\mathfrak{g}$

Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$ be a root space decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. We fix a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset R$

Any disjoint decomposition $\Pi = \Pi^0 \cup \Pi^1$ of Π defines a fundamental gradation of $\mathfrak{g}$ as follows.

We define the function $d : R \to \mathbb{Z}$ by

$$d|_{\Pi^0} = 0,\ d|_{\Pi^1} = 1,\ d(\alpha) = \sum k_i d(\alpha_i),\ \forall \alpha = \sum k_i \alpha_i.$$

Then the fundamental gradation is given by

$$\mathfrak{g}^0 = \mathfrak{h} + \sum_{\alpha \in R,\ d(\alpha)=0} \mathfrak{g}_\alpha\ , \qquad \mathfrak{g}^i = \sum_{\alpha \in R,\ d(\alpha)=i} \mathfrak{g}_\alpha\ .$$

Any fundamental gradation of $\mathfrak{g}$ is conjugated to a unique gradation of such form.

Fundamental gradations of a real semisimple Lie algebra

Any real semisimple Lie algebra $\hat{\mathfrak{g}}$ is a real form of a complex semisimple Lie algebra γ, that is it is the fixed point set $\hat{\mathfrak{g}} = \mathfrak{g}^\sigma$ of some antilinear involution σ of $\mathfrak{g}$, i.e. an antilinear involutive map $\sigma : \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a Lie algebra over $\mathbb{R}$.
We can always assume that σ preserves a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and induces an automorphism of the root system R. A root $\alpha \in R$ is called compact (or black) if $\sigma\alpha = -\alpha$. It is always possible to choose a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ such that, for any non compact root $\alpha_i \in \Pi$, the corresponding root $\sigma\alpha_i$ is a sum of one non-compact root $\alpha_j \in \Pi$ and a linear combination of compact roots from Π. The roots α_i and α_j are called equivalent.

Theorem 15 (Djoković) *Let $\mathfrak{g}$ be a complex semisimple Lie algebra $\mathfrak{g}$, $\sigma : \mathfrak{g} \to \mathfrak{g}$ an antilinear involution and $\mathfrak{g}^\sigma$ the corresponding real form. The gradation of $\mathfrak{g}$, associated with a decomposition $\Pi = \Pi^0 \cup \Pi^1$, defines a gradation $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^i)^\sigma$ of $\mathfrak{g}^\sigma$ if and only if*

Π^1 consists of non compact roots and

any two equivalent roots are either both in Π^0 or both in Π^1.

Example. Fundamental gradations of $\mathfrak{g}_2$

The root system of the complex exceptional Lie algebra $\mathfrak{g}_2$ has the form

$$R = \{\pm\varepsilon_i, \pm(\varepsilon_i - \varepsilon_j),\, i, j =, 2, 3\}$$

where the vectors ε_i satisfy

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,\ \varepsilon_i^2 = 2/3,\ (\varepsilon_i, \varepsilon_j) = -1/3,\ i \neq j\ .$$

Consider the system of simple roots $\Pi = \{\alpha_1 = -\varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3\}$. The corresponding system of positive roots is

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

There are three fundamental gradations for the complex Lie algebra $\mathfrak{g}_2$.

Calculation of the fundamental form $\rho = d\psi$ of a homogeneous para-complex manifold $(M = G/H, K)$ associated with a generalized Gauss decomposition

Now we compute the Koszul form for the homogeneous para-complex manifold
$(M = G^\sigma/H^\sigma, K_M, \text{vol})$,
where M is a covering of an adjoint orbit of a real semisimple Lie group G^σ,
K_M is the invariant para-complex structure defined by a gradation of the Lie algebra $\mathfrak{g}^\sigma$ and vol is an invariant volume form.
First of all, we describe the Koszul form ψ on the Lie algebra

$$\mathfrak{g}^\sigma = \mathfrak{g}^\sigma_{-k} + \cdots + \mathfrak{g}^\sigma_{-1} + \mathfrak{g}^\sigma_0 + \mathfrak{g}^\sigma_1 + \cdots + \mathfrak{g}^\sigma_k \,.$$

or, equivalently, its complex extension ψ to the complex Lie algebra $\mathfrak{g}$ (defined by the same formula).

We choose a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}_0$ of the Lie algebra $\mathfrak{g}$ and denote by R the root system of $(\mathfrak{g}, \mathfrak{a})$. Let

$$\Pi = \Pi^0 \cup \Pi^1, \ \ P = P^0 \cup P^1$$

be the decomposition of a simple root system Π of the root system R which corresponds to the gradation and the corresponding decomposition of the fundamental weights.

Let R^+ be the set of positive roots defined by the basis Π. We put

$$R_0^+ = \{\alpha \in R^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_0\}\,.$$

The following lemma describes the Koszul form in terms of fundamental weights.

Lemma 16 *The* 1*-form* ψ *is equal to*

$$\psi = 2(\delta^{\mathfrak{g}} - \delta^{\mathfrak{h}})$$

where

$$\delta^{\mathfrak{g}} = \sum_{\alpha \in R^+} \alpha\,, \qquad \delta^{\mathfrak{h}} = \sum_{\alpha \in R_0^+} \alpha\,,$$

and the linear forms on the Cartan subalgebra $\mathfrak{a}$ *are considered as linear forms on* $\mathfrak{g}$ *which vanish on root spaces* $\mathfrak{g}_\alpha$*.*

Proposition 17 *Let* $\Pi = \Pi^0 \cup \Pi^1 = \{\alpha_1, \ldots, \alpha_\ell\}$ *be the simple root system (corresponding to the gradation) and denote by* π_i *the fundamental weight corresponding to the simple root* α_i*, namely*

$$2\frac{(\pi_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}\,.$$

If $P^1 = \{\pi_{i_1}, \ldots, \pi_{i_r}\}$*, then the Koszul form* ψ *is equal to*

$$\psi = 2\sum_{\pi \in P^1} n_\pi \pi = 2\sum_{h=1}^{r} a_{i_h}\pi_{i_h}\,, \tag{13}$$

where

$$a_{i_h} = 2 + b_{i_h}\,, \quad \textit{with} \quad b_{i_h} = -2\frac{(\delta^{\mathfrak{h}}, \alpha_{i_h})}{(\alpha_{i_h}, \alpha_{i_h})} \geq 0\,. \tag{14}$$

The main theorem

Theorem 18 *Let R be a root system of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{a}$ and $\mathfrak{g} = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_k$ the fundamental gradation associated with a decomposition $\Pi = \Pi^0 \cup \Pi^1$ of a simple root system $\Pi \subset R$. Let σ be an admissible anti-involution of $\mathfrak{g}$ which defines the graded real form $\mathfrak{g}^\sigma$ of $\mathfrak{g}$ and ψ be the corresponding Koszul form on $\mathfrak{g}$. Let $(M = G^\sigma/H^\sigma, K)$ be a homogeneous manifold of a real semisimple Lie group G^σ with Lie algebra $\mathfrak{g}^\sigma$ such that the stability subalgebra $\mathfrak{h} = \mathfrak{g}_0^\sigma$ and H^σ preserves the generalized Gauss decomposition. Denote by K the invariant para-complex structure on M associated with the gGd and by $\rho = d\psi$ the invariant symplectic form on M defined by $d\psi$. Then for any $\lambda \neq 0$ the pair $(K, \lambda\rho)$ is an invariant para-Kähler Einstein structure on M and this construction exhausts all homogeneous para-Kähler Einstein manifolds of real semisimple Lie groups.*