Lorentzian holonomy groups with applications to parallel spinors

Thomas Leistner, University of Adelaide
Projekt: Geometrische Analysis — Geometry of Lorentzian manifolds with special holonomy

Kolloquium des DFG-Schwerpunktes
Globale Differentialgeometrie
23. – 25. Juli 2004, Universität Leipzig

I. Holonomy groups and parallel spinors

II. Indecomposable, non-irreducible Lorentzian manifolds and their holonomy

III. On the classification of Lorentzian holonomy groups and consequences for parallel spinors

Holonomy groups

Let $(M^{r,s}, h)$ be a semi-Riemannian manifold.

$$\nabla^h \rightsquigarrow \mathcal{P}_\gamma : T_{\gamma(0)}M \xrightarrow{\sim} T_{\gamma(1)}M$$

$$\begin{aligned} Hol_p(M,h) &:= \{\mathcal{P}_\gamma | \gamma \text{ loop in } p\} \subset O(T_pM, h_p) \\ Hol_p^0(M,h) &:= \{\mathcal{P}_\gamma | \gamma \sim p\} \subset SO_0(T_pM, h_p) \end{aligned}$$

Lie group with Lie algebra $\mathfrak{hol}_p(M,h)$, closed if it acts completely reducible.

Parallel spinors

$(M^{r,s}, h)$ spin $\rightsquigarrow$ spin bundle S, ∇^S lift of ∇^h.

$$\{\varphi \in \Gamma(S) \mid \nabla^S \varphi = 0\}$$

$$\updownarrow$$

$$\left\{ \text{ spinors fixed under } \widetilde{Hol}_p(M,h) \subset Spin(r,s) \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{annihilated spinors under} \\ \text{the spin representation of } \mathfrak{hol}_p(M,h) \end{array} \right\}$$

Question: What are the holonomy groups of manifolds with parallel spinors?

Decomposition

– **Holonomy** [de Rham'52, Wu'64]:
(M, h) simply connected, complete $\Rightarrow$

$$Hol(M,h) = H_1 \times \ldots \times H_k \iff$$
$$(M,h) \overset{\text{isom.}}{\simeq} (M_1,h_1) \times \ldots \times (M_k,h_k),$$

(M_i, h_i) simply connected, complete,
$H_i = Hol(M_i, h_i)$ trivial or indecomposable
(:= no *non-degenerate* invariant subspace)

– **Parallel spinors:**

Let $H_1 \subset SO(r_1, s_1)$, $H_2 \subset SO(r_2, s_2)$. Then:

$\widetilde{H_i} \subset Spin(r_i, s_i)$ fix a spinor $\iff$
$\widetilde{H_1 \times H_2} \subset Spin(r_1 + r_2, s_1 + s_2)$ fixes a spinor.

$\rightsquigarrow$ $(M_1, h_1) \times (M_2, h_2)$ has parallel spinor fields $\iff$ (M_1, h_1) and (M_2, h_2) have parallel spinor fields.

Question: What are indecomposable holonomy groups of manifolds with parallel spinors?

Berger algebras

Ambrose-Singer theorem + Bianchi-identity
$\rightsquigarrow$ algebraic constraints on $\mathfrak{hol}_p(M,h) \subset \mathfrak{so}(T_pM,h)$:

Let $\mathfrak{g} \subset \mathfrak{gl}(E)$ be a Lie algebra.

- $\mathcal{K}(\mathfrak{g}) :=$
$\{R \in \Lambda^2 E^* \otimes \mathfrak{g} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0\}$

- $\underline{\mathfrak{g}} \ := \ span\{R(x,y) \mid x,y \in E, R \in \mathcal{K}(\mathfrak{g})\}$

$\mathfrak{g} \subset \mathfrak{gl}(E)$ is called **Berger algebra** $\overset{def.}{\Longleftrightarrow}$ $\underline{\mathfrak{g}} = \mathfrak{g}$.

Ambrose-Singer $\Rightarrow \mathfrak{hol}_p(M,h)$ is a Berger algebra.

Classification of irreducible Berger algebras:
– $\mathfrak{g} \subset \mathfrak{so}(r,s)$ [Berger '55] and
– $\mathfrak{g} \subset \mathfrak{gl}(n)$ [Schwachhöfer/Merkulov '99]
$\rightsquigarrow$ list of holonomy groups of (simply connected) **irreducible** semi-Riemannian/affine manifolds.

Berger-list of irreducible holonomy groups

Riemannian: [Berger '55, Simons '62, ...]
$SO(m)$, $U(p)$, $SU(p)$, $Sp(q)$, $Sp(1) \cdot Sp(q)$, $Spin(7)$, G_2 and holonomy groups of symmetric spaces.

Lorentzian: only $SO_0(1, m-1)$!
(Direct proof by Olmos/diScala '01)

$\rightsquigarrow$ **Irreducible holonomy groups of semi-Riemannian manifolds with parallel spinors:**

Riem.: [Wang '89] $SU(p)$, $Sp(q)$, $Spin(7)$, G_2

pseudo-Riem.: [Baum/Kath '99]
$SU(p,q)$, $Sp(p,q)$, $G^*_{2(2)} \subset SO(7,7)$, $G_2^{\mathbb{C}} \subset SO(7,7)$, $Spin(4,3) \subset SO(4,4)$, $Spin(7)^{\mathbb{C}} \subset SO(8,8)$.

No irreducible Lorentzian manifolds with parallel spinors!

Riemannian: indecomposable = irreducible

$\Longrightarrow$ de Rham-decomposition + Berger-list

$\rightsquigarrow$ Classification of all holonomy groups of simply-connected, complete Riemannian manifolds (with parallel spinors).

Simply connected, complete Lorentzian mfds.

Indecomposable $\neq$ irreducible. Wu-decomposition:

$$
\begin{array}{ccccccccc}
(M,h) \simeq & (\widehat{M},\widehat{h}) & \times & (N_1,g_1) & \times \ldots \times & (N_k,g_k) \\
 & \uparrow & & \uparrow & \uparrow & \uparrow
\end{array}
$$

<u>Lorentzian</u>: Riemannian, irred. or flat

- flat ($\checkmark$)
- irred., i.e. $Hol = SO_0(1,m-1)$ ($\checkmark$)
- indecomposable, non-irreducible, i.e. with degenerate, holonomy -invariant subspace

Parallel spinors on Lorentzian manifolds

- Spinorfield $\varphi \rightsquigarrow$ **causal** vector field V_φ via:

$$h(V_\varphi, X) = -\langle X \cdot \varphi, \varphi \rangle_S .$$

- φ **parallel** $\Longrightarrow$ V_φ parallel.

$\rightsquigarrow$ 2 cases —

<u>V_φ timelike:</u>

$$\begin{array}{cccccc} (M,h) \simeq & (\mathbb{R}, -dt^2) & \times & (N_1, g_1) & \times \ldots \times & (N_k, g_k) \\ & & & \uparrow & \uparrow & \uparrow \end{array}$$

Riemannian with
$Hol = SU(k), Sp(k), G_2, Spin(7)$

<u>V_φ lightlike:</u>

$$\begin{array}{cccccc} & & & \downarrow & \downarrow & \downarrow \\ (M,h) \simeq & (\widehat{M}, \widehat{h}) & \times & (N_1, g_1) & \times \ldots \times & (N_k, g_k) \\ & \uparrow & & & & \end{array}$$

indecomposable Lorentzian
with parallel lightlike vector field

II. Lorentzian manifolds (M^{n+2}, h) with indecomposable, non-irreducible holonomy

$E \subset T_pM$ be degenerate, holonomy invariant
$\Longrightarrow$ $L = E \cap E^{\perp} \subset T_pM$ is one-dimensional, lightlike and holonomy invariant.
$\Longrightarrow$ $Hol_p(M,h) \subset Iso(L) \subset SO(T_pM, h_p)$.

$L \longleftrightarrow$ recurrent, lightlike vector field X

$$\mathfrak{hol} \subset \underbrace{\mathfrak{iso}(L) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n}_{\left\| \atop \left\{ \left(\begin{array}{ccc} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{array} \right) \middle| \begin{array}{l} a \in \mathbb{R}, \\ v \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\} \right.} \subset \mathfrak{so}(1, n+1).$$

$\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{hol}$ — "screen holonomy"

---

parallel spinor $\Longleftrightarrow$
$pr_{\mathbb{R}}(\mathfrak{hol}) = 0$ and $\exists\ v \in \Delta_n\ :\ \mathfrak{g}v = 0$.
In fact:
$\dim\{\text{par. spinors}\} = \dim\{v \in \Delta_n | \mathfrak{g}v = 0\} \leq 2^{[\frac{n}{2}]}$

Properties:

$\mathfrak{h} = \mathfrak{hol}_p(M,h) \subset \mathfrak{iso}(L)$ indecomposable. $\Longrightarrow$

- $pr_{\mathbb{R}^n}(\mathfrak{h}) = \mathbb{R}^n$.

- $\mathfrak{h}$ is Abelian $\iff$ $\mathfrak{h} = \mathbb{R}^n$.

- $pr_{\mathbb{R}}\mathfrak{h} = 0$ $\iff$ $\exists$ parallel lightlike vectorfield.

- $\mathfrak{g} = pr_{\mathfrak{so}(n)}(\mathfrak{h})$ is compact and therefore reductive, i.e. $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{d}$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{d} = [\mathfrak{g}, \mathfrak{g}]$.

Indecomposable $\Longrightarrow$

parallel spinor $\rightsquigarrow$ *lightlike* parallel vector field
$\rightsquigarrow$ $\mathfrak{hol}(M,h) \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$.

(M,h) indecomposable with $\mathfrak{hol} \subset \mathfrak{so}(n) \ltimes \mathbb{R}^n$ admits a parallel spinor $\iff$ spin representation of $\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{hol}$ has trivial subrepresentation.
In fact:
Dim {parallel spinors} = Dim $\{v \in \Delta_n | \mathfrak{g}v = 0\} \leq 2^{[\frac{n}{2}]}$

Question: Which Lie algebras occur as

$$\mathfrak{g} = pr_{\mathfrak{so}(n)}(M,h)$$

and which of these has trivial spin subrepresentations?

Four algebraic types [Berard-Bergery/Ikemakhen'93]

$\mathfrak{h} \subset \mathfrak{iso}(L) \subset \mathfrak{so}(1, n+1)$ indecomposable,

$\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h}) = \mathfrak{z} \oplus \mathfrak{d}$.

Then $\mathfrak{h}$ belongs to one of the following types:

1st case — $\mathfrak{h}$ contains $\mathbb{R}^n$:

Type 1: $\mathfrak{h}$ contains $\mathbb{R}$, i.e. $\boxed{\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n}$.

Type 2: $pr_{\mathbb{R}}(\mathfrak{h}) = 0$ i.e. $\boxed{\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n}$.

Type 3: Neither Type 1 nor Type 2.

$\exists\ \varphi : \mathfrak{z} \to \mathbb{R}$ surjective:

$$\boxed{\mathfrak{h} = \left\{ \left(\begin{array}{ccc} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^n \right\}}.$$

2nd case — $\mathfrak{h}$ does not contain $\mathbb{R}^n$ — **Type 4:**

$\exists$ a decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, $0 < k, l < n$,
$\exists\ \varphi : \mathfrak{z} \to \mathbb{R}^l$ surjective:

$$\boxed{\mathfrak{h} = \left\{ \left(\begin{array}{cccc} 0 & \varphi(A)^t & v^t & 0 \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & -\varphi(A) \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^k \right\}}.$$

Indecomposable, non-irreducible Lorentzian manifolds with Abelian holonomy

Example: Symmetric spaces with solvable transvection group (Cahen-Wallach spaces).

Proposition. ['00] (M, h) has Abelian holonomy $\mathbb{R}^n \iff$ it is an indecomposable pp-wave.

Def. A Lorentzian mfd. is called ***pp*-wave** $:\iff \exists$ parallel lightlike vector field and:

$$tr_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0.$$

In coordinates:

$$h = dxdz + f dz^2 + \sum_{i=1}^{n} dy_i^2, \text{ with } \frac{\partial f}{\partial x} = 0.$$

($\exists$ a similar result for $\mathfrak{hol}(M, h) = \mathbb{R} \ltimes \mathbb{R}^n$.)

$n+2$-dim. pp-wave admits $2^{[\frac{n}{2}]}$ parallel spinors which are pure, i.e.
$dim\{Z \in TM^{\mathbb{C}} | Z \cdot \varphi = 0\} = \frac{dim\ M}{2}$

Construction of indecomposable, non-irreducible Lorentzian manifolds with

$pr_{SO(n)} Hol(M,h) =$ **Riemannian holonomy**

Proposition. ['00] Let
- (N,g) be a Riemannian manifold,
- γ be a function of $z \in \mathbb{R}$,
- $f \in C^\infty(N \times \mathbb{R}^2)$, sufficiently generic,
- ϕ_z be a family of 1–forms on N, such that

$$d\phi_z \in \mathcal{H}(N,g) \times_{Hol_p(N,g)} \mathfrak{hol}_p(N,g).$$

$\Longrightarrow$ The Lorentzian manifold

$$\Big(M = N \times \mathbb{R}^2, h = 2dxdz + fdz^2 + \phi_z dz + e^{2\gamma} \cdot g\Big)$$

is indecomposable, non-irreducible and

$$pr_{SO(n)} Hol_{(x,p,z)}(M,h) = Hol_p(N,g).$$

(N,g) parallel spinors $\Rightarrow$ (M,h) parallel spinors.

$Hol(N,g) = \{1\}$, $SU(p)$, $Sp(q)$ $\Rightarrow$ pure.

Coordinates for indecomposable, non-irreducible Lorentzian manifolds: [Walker'49, Brinkmann'25]

On (M^{n+2}, h) $\exists$ **recurrent**, lightlike vector field $\iff$ $\exists$ coordinates $(U, \varphi = (x, (y_i)_{i=1}^n, z))$:

$$h = 2\ dxdz + \underbrace{\sum_{i=1}^{n} u_i dy_i\, dz}_{:=\phi_z} + f dz^2 + \underbrace{\sum_{i,j=1}^{n} g_{ij} dy_i\ dy_j}_{g_z}$$

with $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$, $f \in C^\infty(M)$.

$\exists$ **parallel** lightlike vector field $\iff$ $\frac{\partial f}{\partial x} = 0$.

$(U, \varphi) \rightsquigarrow$ family of n-dimensional Riemannian submanifolds with metric g_z and 1-form ϕ_z:

$$W_{(x,z)} := \left\{ \varphi^{-1}(x, y_1, \ldots, y_n, z) | (y_1, \ldots, y_n) \in \mathbb{R}^n \right\}$$

Proposition. [Ikemakhen '96]

$$Hol_p(W_{x(p),z(p)}, g_{z(p)}) \ \subset\ pr_{SO(n)} Hol_p(M, h).$$

Question: Under which conditions "$=$"?

III. On the classification of Lorentzian holonomy groups

Result: ['02, '03] $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}(M,h)$ for (M,h) indecomposable, non-irred. Lorentzian $\Longleftrightarrow$ $\mathfrak{g}$ is a Riemannian holonomy algebra.

Consequence: $\mathfrak{h} = \mathfrak{hol}_p(M,h) \Longrightarrow$

<u>Type 1 or 2</u>: $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$ or $\mathfrak{g} \ltimes \mathbb{R}^n$ with $\mathfrak{g} =$ Riemannian holonomy algebra.

<u>Type 3 or 4</u>: $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{h}$ is a Riemannian holonomy algebra with center, i.e. with $\mathfrak{so}(2)$ summand.

Corollary. ['03] $H =$ holonomy group of an indecomposable Lor. mf. with **parallel spinors**. $\Longrightarrow$ H is of uncoupled type 2, i.e.

$$H = G \ltimes \mathbb{R}^n,$$

$G =$ product of $\{1\}$, $SU(p)$, $Sp(q)$, G_2 or $Spin(7)$.

This generalizes a result of R. Bryant ['99] for $n \leq 9$.

Proof is by algebraic means, direct geometric proof is desirable.

Up to dim 9 proved also by Galaev '03, also algebraically.

Remark on coupled types 3 and 4: $\mathfrak{g}$ = Riemannian holonomy with center
$\Longrightarrow$ it is a sum of the following:

- $\mathfrak{so}(2)$ acting on $\mathbb{R}^2$ or on itself,
- $\mathfrak{so}(2) \oplus \mathfrak{so}(n)$ acting on $\mathbb{R}^{2n}$,
- $\mathfrak{so}(2) \oplus \mathfrak{so}(10)$ acting on $\mathbb{R}^{32}$ = reellification of the cx. spinor module of dimension 16,
- $\mathfrak{so}(2) \oplus \mathfrak{e}_6$ acting on $\mathbb{R}^{54}$,
- $\mathfrak{u}(n)$ acting on $\mathbb{R}^{2n}$ or on $\mathbb{R}^{n(n-1)}$.

Then apply method of Ch. Boubel ['00]:

(M, h) with holonomy of uncoupled type 1 or 2, $\mathfrak{g}$ with center (+further algebraic constraints)
$\rightsquigarrow$ construct locally a metric on M with holonomy of coupled type 3 or 4.

To do this: *Start with entries of the above list!*

Proof. (of the corollary) Type 4 $\Rightarrow$ G = Riemannian holonomy with center has $SO(2)$-factor, spin representations of $SO(2)$ has no fixed vectors. $(*)$ $\Rightarrow$ Proposition. $\square$

Proof: 1. Weak-Berger algebras

Let (E, h) be a Euclidean vector space $\mathfrak{g} \subset \mathfrak{so}(E, h)$:

$\mathcal{B}_h(\mathfrak{g}) := \{Q \in Hom(E, \mathfrak{g}) \mid$

$h(Q(x)y, z) + h(Q(y)z, x) + h(Q(z)x, y) = 0\}$

$$\mathfrak{g}_h := span\{Q(x) \mid x \in E, Q \in \mathcal{B}_h(\mathfrak{g})\}.$$

$\mathfrak{g} \subset \mathfrak{so}(E, h)$ **weak-Berger algebra** $\Leftrightarrow \mathfrak{g}_h = \mathfrak{g}$.

Lemma. Orthogonal Berger $\Longrightarrow$ weak-Berger.
Because: $\mathcal{R} \in \mathcal{K}(\mathfrak{g}) \implies \mathcal{R}(x, .) \in \mathcal{B}_h(\mathfrak{g})$

Decomposition:
h positive definite $\Rightarrow E = E_0 \oplus \ldots \oplus E_r$:
$\mathfrak{g}$ trivial on E_0, irred. on E_i, $i \geq 1$.

$\mathfrak{g}$ weak-Berger $\Rightarrow \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$:
$\mathfrak{g}_i$ ideals, irred. on E_i and trivial on E_j for $i \neq j$,
$\mathfrak{g}_i \in \mathfrak{so}(E_i, h|E_i)$ weak-Berger.

Problem:

$\mathfrak{hol}$ is a Berger algebra but not completely reducible.

$\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}$ is completely reducible, but not a holonomy algebra and hence not a Berger algebra, apriori.

$\leadsto$ weak-Berger algebras.

2. Weak-Berger algebras and Lorentzian holonomy

Prop. ['02] Let (M^{n+2}, h) be indecomposable, non-irreducible Lorentzian, $\mathfrak{g} = pr_{\mathfrak{so}(n)}(\mathfrak{hol}_p(M,h))$. Then $\mathfrak{g}$ is a weak-Berger algebra.

In particular: $\mathfrak{g}$ decomposes into irreducibly acting ideals [Berard-Bergery, Ikemakhen '93] which are weak Berger.

Corollary. [Berard-Bergery, Ikemakhen '93]
$G := pr_{SO(n)} Hol_p(M,h)$ is closed in $SO(n)$ and therefore compact.
☛ This is not true for the whole holonomy group, there are non-closed examples!

$\rightsquigarrow$ Classify real, irreducible weak-Berger algebras!

Proof. (of the proposition) Bianchi-identity restricted to the non-degenerate subspace $\mathbb{R}^n \subset T_pM$ and AS $\rightsquigarrow$

$$\mathfrak{g} = span \begin{pmatrix} \{R(u,v) | R \in \mathcal{K}(\mathfrak{g}), u, v \in E\} \\ \cup \ \{Q(w) | Q \in \mathcal{B}_h(\mathfrak{g}), w \in E\} \end{pmatrix} \subset \mathfrak{g}_h.$$

$Q \ \widehat{=} \ pr_E \ \circ \mathcal{R}(.,Z)_{|E \times E}$ with Z lightlike and transversal to $L_p^{\perp}$ $\square$

3. Transition to the complex situation

E real, $\mathfrak{g} \subset \mathfrak{so}(E, h) \implies (\mathcal{B}_h(\mathfrak{g}))^{\mathbb{C}} = \mathcal{B}_{h^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}})$.

Lemma. $\mathfrak{g} \subset \mathfrak{so}(E, h)$ weak-Berger $\iff$
$\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$ weak-Berger.

$\mathfrak{g} \subset \mathfrak{so}(E, h)$ irreducible $\implies$

1. $\underline{E^{\mathbb{C}} \text{ irreducible}}$:
$\Rightarrow \mathfrak{g}$ semisimple
$\rightsquigarrow$ weak-Berger property in terms of roots and weights
$\rightsquigarrow$ result for simple and semisimple Lie algebras (uses ideas of Schwachhöfer '99).

2. $\underline{E^{\mathbb{C}} \text{ not irreducible}}$:
$\Rightarrow E^{\mathbb{C}} = V \oplus \overline{V}$, V irreducible, $\mathfrak{g} \subset \mathfrak{u}(V) \Rightarrow$

$$(\mathcal{B}_h(\mathfrak{g}))^{\mathbb{C}} \simeq \text{ first prolongation of } \mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(V).$$

Classification of irred. acting complex Lie algebras with non-vanishing first prolongation by Kobayashi/Nagano ['65]
$\rightsquigarrow$ result for unitary Lie algebras

4. Weak-Berger algebras of real type

Let $\mathfrak{g} \subset \mathfrak{so}(V,H)$ irreducible, semisimple complex LA.

- Δ_0 := roots and zero of $\mathfrak{g} = \bigoplus_{\alpha\in\Delta_0} \mathfrak{g}_\alpha$
- Ω := weights of $V = \bigoplus_{\mu\in\Omega} V_\mu$.
- Π := weights of $\mathcal{B}_H(\mathfrak{g}) = \bigoplus_{\phi\in\Pi} \mathcal{B}_\phi$.
- $Q \in \mathcal{B}_\phi$ and $v \in V_\mu \implies Q(u) \in \mathfrak{g}_{\phi+\mu}$

- $$\Gamma := \left\{ \mu + \phi \;\middle|\; \begin{array}{l} \mu \in \Omega,\ \phi \in \Pi \quad \exists u \in V_\mu \\ \text{and } Q \in \mathcal{B}_\phi\text{: } Q(u) \neq 0 \end{array} \right\} \subset \Delta_0 \subset \mathfrak{t}^*.$$

Prop. If $\mathfrak{g} \subset \mathfrak{so}(V,h)$ is weak-Berger, then $\Gamma = \Delta_0$.

Lemma: If $\mathcal{B}_H(\mathfrak{g}) \neq 0$, $\Lambda \in \Omega$ extremal $\Rightarrow$
$\forall\ u \in V_\Lambda\ \exists\ Q \in \mathcal{B}_\phi$: $Q(u) \neq 0$.

Define for $\alpha \in \Delta$ the weights of $\mathfrak{g}_\alpha V$:

$$\Omega_\alpha := \{\mu \in \Omega \mid \mu + \alpha \in \Omega\}.$$

Proposition. ['03] Let $\mathfrak{g} \subset \mathfrak{so}(V,H)$ as above. Then:

$\exists$ extremal weight Λ, hyperplane $U \subset \mathfrak{t}^*$, $\alpha \in \Delta$:

(1) $\Omega \subset \{\Lambda + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}$.

or

(2) $\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}$.

Checking the representations of simple Lie algebra whether they obey (1) or (2)
$\rightsquigarrow$ Simple weak-Berger algebras of real type are complexifications of Riemannian holonomy algebras.

For semisimple Lie algebras:

V irreducible $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-module $\Longrightarrow V = V_1 \otimes V_2$ where V_i are irreducible $\mathfrak{g}_i$-modules.

$\rightsquigarrow$ derive criteria to the summands $\mathfrak{g}_i \subset \mathfrak{gl}(V_i)$

$\rightsquigarrow$ semisimple weak-Berger algebras are complexifications of Riemannian holonomy algebras

5. Weak-Berger algebras of non-real type

Let $\mathfrak{g}_0 \subset \mathfrak{so}(E, h)$ be real, but of non-real type. $\Longrightarrow$

- $E^{\mathbb{C}} = V \oplus \overline{V}$ $\mathfrak{g}_0$-invariant, $E = V_{\mathbb{R}}$,
- $(\mathfrak{g}_0)\Big|_V$ is irreducible and unitary, but not orthogonal.

$\mathfrak{g} := (\mathfrak{g}_0^{\mathbb{C}})\Big|_V$ — define the **first prolongation**

$$\mathfrak{g}^{(1)} := \{Q \in Hom(V, \mathfrak{g}) \mid Q(u)v = Q(v)u\}.$$

Proposition. ['02] There is an isomorphism

$$\begin{array}{rcccl} \phi & : & \mathcal{B}_H(\mathfrak{g}_0^{\mathbb{C}}) & \simeq & \mathfrak{g}^{(1)} \\ & & Q & \mapsto & Q_{|V \times V}. \end{array}$$

Weak-Berger $\Longrightarrow \mathcal{B}_H(\mathfrak{g}) \neq 0$

Classification of Lie algebras with non-trivial 1. prolongation [Kobayashi/Nagano '65]

Table 1 Complex Lie algebras with $\mathfrak{g}^{(1)} \neq 0$, $\mathfrak{g}^{(1)} \neq V^*$:

$\mathfrak{g}$	V	$\mathfrak{g}^{(1)}$	Riem. Hol.
$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n, \quad n \geq 2$	$(V \otimes \odot^2 V^*)_0$	$\mathfrak{su}(n)$
$\mathfrak{gl}(n,\mathbb{C})$	$\mathbb{C}^n, \quad n \geq 1$	$V \otimes \odot^2 V^*$	$\mathfrak{u}(n)$
$\mathfrak{sp}(n,\mathbb{C})$	$\mathbb{C}^{2n}, \quad n \geq 2$	$\odot^3 V^*$	$\mathfrak{sp}(n)$
$\mathbb{C} \oplus \mathfrak{sp}(n,\mathbb{C})$,	$\mathbb{C}^{2n}, \quad n \geq 2$	$\odot^3 V^*$	not w-B!

Table 2 Complex Lie algebras with $\mathfrak{g}^{(1)} = V^*$:

$\mathfrak{g}$	V	Riem. symm. space
$\mathfrak{co}(n,\mathbb{C})$	$\mathbb{C}^n$,	$SO(2+n)/SO(2) \cdot SO(n)$
$\mathfrak{gl}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$	$Sp(n)/U(n)$
$\mathfrak{gl}(n,\mathbb{C})$	$\wedge^2\mathbb{C}^n$	$SO(2n)/U(n)$
$\mathfrak{sl}(\mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{gl}(m,\mathbb{C}))$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$SU(n+m)/U(n) \cdot U(m)$
$\mathbb{C} \oplus \mathfrak{spin}(10,\mathbb{C})$	$\Delta_{10}^+ \simeq \mathbb{C}^{16}$	$E_6/SO(2) \cdot Spin(10)$
$\mathbb{C} \oplus \mathfrak{e}_6$	$\mathbb{C}^{27}$	$E_7/SO(2) \cdot E_6$

$\rightsquigarrow$ every irreducible weak-Berger algebra which is unitary is a Riemannian holonomy algebra.

Decomposition property
$\rightsquigarrow$ every unitary weak-Berger algebra is a Riemannian holonomy algebra.