Kapitel 2

Hyperbolische Erhaltungsgleichungen

In diesem Kapitel werden wir hyperbolische Erhaltungsgleichungen studieren. Ein Beispiel für die diesen Typ kennen wir bereits die Burgers Gleichung. Sie hat uns gezeigt, dass man den Lösungsbegriff erweitern sollte, da klassische Lösungen eventuell nur auf kleinen Teilmengen existieren und wichtige Aspekte dabei ausgelassen werden.

Inhaltsangabe

2.1 Hyperbolische Gleichungen

Many problems in physics can be derived from fundamental principles like conservation laws. Well known such conservation laws are conservation of mass, conservation of energy, etc.. Conservation laws plays an important rôle in the study of fluid mechanics and in the development of shock waves and similar problems. We will state the underlying problem, point out how the methods of characteristics can be used, as in the study of Burgers' equation to prove the nonexistence of continuous solutions. Then we introduce the notion of weak solutions and the Rankine-Hugoniot condition, which connect shock speed with the height of the jump across a discontinuity. Then we discuss the notion of entropy solution for such equations. In this context we address the question of physically meaningful solutions. This discussion ends with the notion of viscosity solution and its relation to travelling waves and ODE's.

2.1.1 Conservation Laws

We begin with the local form of conservation laws, where we have a (nonlinear) map $f : \mathbb{R}^n \to \mathbb{R}^n$ and we consider the partial differential equation

$$u_t + (f(u))_x = 0 \tag{2.1}$$

for an unknown function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$. In higher dimensional spaces (for the space variable x), the theory becomes much more complicated, we restrict ourselves to the case $x \in \mathbb{R}$.
We have already seen a well known example. Burgers equation is given by

$$u_t + uu_x = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ takes the form

$$f(u) = \frac{1}{2}u^2.$$

For a discussion of the relation between global and local forms of conservation laws consult consult for example Evans [7].

Beispiel 2.1.1
We consider the following simple experiment, a one dimensional unbounded domain is supposed to be filled with gas and separated into two compartments by a membrane (wlog at $x = 0$). Suppose on one side of the membrane ($x < 0$) we have constant average speed $u = 0$, constant density $\rho = \rho_l$ and constant pressure $p = p_l$, on the right side $x > 0$ we have $u = 0$, $\rho = \rho_r$ and $p = p_r$. After removing the membrane we will see a wave moving from the higher to the lower density. In the back of this wave we have a so called rarefaction wave, a wave of lower density.

2.1.2 Weak Solutions, Shocks and the Rankine-Hugoniot condition

Before we give a technical definition of weak solutions we briefly recall Gauß or divergence theorem and point out how it relates to integration by parts in higher dimensional spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain with a sufficiently regular boundary and let $u : \Omega \to \mathbb{R}^n$ be a vectorfield, then we have

$$\int\limits_{\Omega} \operatorname{div} u \mathrm{d}\lambda = \int\limits_{\partial\Omega} \langle n, u \rangle \mathrm{d}\omega$$

where n denotes the outward unit normal to $\partial\Omega$, λ denotes the n-dimensional Lebesgue measure and ω is the Lebesgue measure on $\partial\Omega$. If u, v are C^1-functions on Ω and e_i is the unit vector with jth-entry δ_{ij} then we consider the vectorfield uve_i and obtain

$$\int\limits_{\Omega} u_{x_i} v + u v_{x_i} \mathrm{d}\lambda = \int\limits_{\partial\Omega} u v n_i d\omega$$

where n_i is the i-th component of n. If we consider Ω to be a rectangular box whose bounding planes are parallel to the standard hyperplanes $x_j = 0$ in $\mathbb{R}^n$, ie when Ω has the form

$$\Omega = [a_1, b_1] \times \cdots \times [a_i, b_i] \times \cdots \times [a_n, b_n]$$

then we get the special formula:

$$. \int\limits_{\Omega} u_{x_i} v \mathrm{d}\lambda = -\int\limits_{\Omega} u v_{x_i} d\lambda + \int\limits_{\Omega'} \left((uv)(x_1, \ldots, b_i, \ldots, x_n) - (uv)(x_1, \ldots, a_i, \ldots, x_n) \right) d\lambda_{n-1}, \tag{2.2}$$

where

$$(uv)(x_1, \ldots, *, \ldots, x_n) = u(x_1, \ldots, *, \ldots, x_n) v(x_1, \ldots, *, \ldots, x_n)$$

and $d\lambda_{n-1}$ is the $n-1$-dimensional Lebesgue measure.

Now we can define the notion of a *weak solutions* for conservation laws. In order to do so we consider functions $\varphi \in C^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$ with compact support, ie

$$\varphi : [0, \infty) \times \mathbb{R} \to \mathbb{R}^n$$

with

$$\operatorname{supp}\varphi \subset [0,T) \times K^0$$

where $0 < T < \infty$ and $K \subset \mathbb{R}$ is compact and K^0 is the interior of K. Let $C_0^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^)$ denote the space of all these functions and assume $u \in C^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$ is a solution of the conservation law

$$u_t + (f(u))_x = 0.$$

Then we have the following property where the integral is to be understood component wise and we use the formula which we have just looked at:

$$\begin{aligned}
0 &= \int\limits_{\mathbb{R}^+ \times \mathbb{R}} (u_t + \nabla_x f(u))\,\varphi\, d(x,t) \\
&= \int\limits_0^T \int\limits_K (u_t + (f(u))_x)\varphi\; dx\; dt \\
&= \int\limits_0^T \int\limits_K u_t\varphi\; dxdt + \int\limits_0^T \int\limits_K f(u)_x\varphi\; dx\; dt \\
&= \int\limits_K \int\limits_0^T u_t\varphi\; dtdx + \int\limits_0^T \int\limits_K f(u)_x\varphi\; dx\; dt \\
&= \int\limits_K u\varphi\Big|_0^T dx - \int\limits_K \int\limits_0^T u\varphi_t\; dt\; dx + \int\limits_0^T \int\limits_K f(u)_x\varphi\; dx\; dt \\
&= -\int\limits_K u_0(x)\varphi(0,x)\; dx - \int\limits_0^T \int\limits_K u\varphi_t - f(u)_x\varphi\; dx\; dt.
\end{aligned}$$

We obtain

$$0 = -\int\limits_K u_0(x)\varphi(0,x)\; dx - \int\limits_0^T \int\limits_K u\varphi_t + f(u)\varphi_x\; dx\; dt. \tag{2.3}$$

Definition 2.1.2
A bounded measurable function $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ is called a weak solution *of the conservation law (2.1) if Equation (2.3) is satisfied for all $\varphi \in C_0^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$.*

Lemma 2.1.3
A function $u \in C^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^n)$ is a weak solution of Equation (2.1) if and only if it is a solution.

Beweis. This is a simple observation: we can do the foregoing computation backward and we obtain for a C^1 weak solution

$$\int\limits_{\mathbb{R}^+ \times \mathbb{R}} (u_t + (f(u))_x)\varphi = 0$$

for all functions $\varphi \in C_0^1(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. This implies that

$$u_t + (f(u))_x = 0.$$

□

Bemerkung 2.1.4
We see that the notion of a weak solution is a true generalization of the notion of a solution. As we shall see later it is obviously related to a similar notion in the context of elliptic equations.

Let us briefly discuss the behavior of solutions of Burgers' equation: given an initial value

$$u_0 : \mathbb{R} \to \mathbb{R}$$

we want to construct a solution. In order to do this we look at level sets of of u, ie, we consider

$$L(c) = \left\{ (x,t) \;\middle|\; u(x,t) = c \right\}.$$

Then $\nabla_{(x,t)} u$ is orthogonal to the tangent vector of $L(c)$ at (t_0, x_0). Due to the differential equation the vector $(1, u)$ is orthogonal to $\nabla_{(x,t)} u$ and hence is tangent to $L(c)$ at (t_0, x_0). Since this direction $(1, c)$ is independent of (t_0, x_0) the set $L(c)$ is a line, ie

$$L(c) = \left\{ (t_0 + t, x_0 + ut) \;\middle|\; t \geq -t_0 \right\}.$$

This implies that the initial data gives rise to a unique solution which is as smooth as the initial data as long as different level sets do not intersect. Consider the continuous initial data

$$u_0(x) = \begin{cases} 1 & , \quad x \leq 0 \\ 1-x & , \quad x \in [0,1] \\ 0 & , \quad x \geq 1. \end{cases}$$

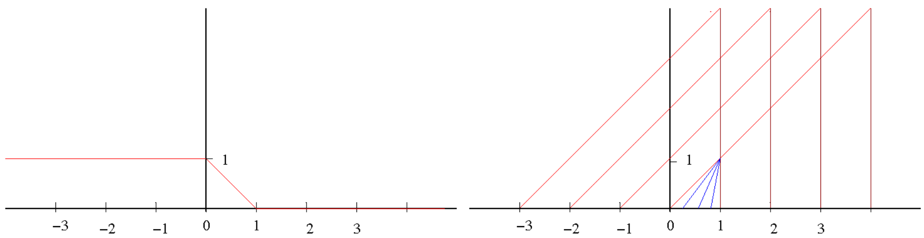
Then $u(x,t) = 1$ on $x \leq t$. It is equal to 0 on $x \geq 1$. Obviously these two sets intersect and hence we have a formation of discontinuities. The following graphic shows the (x,t)-plane and the level sets of the solution of Burgers' equation with the given initial data $u_0(x)$. In order to get beyond this point we need a new, more general notion of solution.

Aufgabe 2.1.5
Show that the classical solution of Burgers' equation corresponding to the above initial value is defined on

$$[0,1) \times \mathbb{R}$$

and it has the form

$$u_1(x,t) = \begin{cases} 1 & , \quad x < t < 1 \\ \frac{1-x}{1-t} & , \quad t \leq x \leq 1, t < 1 \\ 0 & , \quad x > 1, t < 1 \end{cases} . \tag{2.4}$$



Lemma 2.1.6
Burgers equation with the initial condition $u_0(x_0)$ defined above possesses a weak solution. It is given by

$$u(x,t) = \begin{cases} u_1(x,t) & , \quad 0 \leq t < 1 \\ u_2(x,t) & , \quad 1 \leq t < \infty, \end{cases} \tag{2.5}$$

where u_1 is the solution described before in Problem 2.1.5, and u_2 is given by

$$u_2(x,t) = \begin{cases} 0 & , \quad x < \frac{1+t}{2} & , \quad t > 1 \\ 1 & , \quad x > \frac{1+t}{2} & , \quad t > 1. \end{cases}$$

In this example we have the development of a shock, with a certain shock speed ($\frac{1}{2}$ in this case) and a height of the jump. Instead of proving this lemma we give a more general result which describes a general relation between the shock speed and the height of the jump. This condition relates to the so called Rankine[1]- Hugoniot[2] condition in gas dynamics.

Satz 2.1.7
Let $u(x,t)$ be a weak solution of a conservation law

$$u_t + (f(u))_x = 0$$

and assume we have a curve $C(t) = (t, x(t))$ of discontinuities of u. We assume that C is differentiable and it separates our domain into two open sets, which we call the left set U_l and the right set U_r. We assume that at each point on C we have a left and a right limit, ie

$$u_l(x_0,t_0) = \lim_{(x,t)\to(x_0,t_0),(x,t)\in U_l} u(x,t)$$

and

$$u_r(x_0,t_0) = \lim_{(x,t)\to(x_0,t_0),(x,t)\in U_r} u(x,t)$$

exist. We call $s = x'(t)$ the speed of the discontinuity at $(t, x(t))$. Then we have the fundamental relation

$$s(u_r - u_l) = f(u_r) - f(u_l).$$

This is called Rankine-Hugoniot condition

Beweis. Let (x_0, t_0) be a point on the curve C and D be a small rectangular domain centered at (x_0, t_0) and let $\varphi \in C_0^1(D)$ be a C^1 function with compact support in D. Observe that the curve of discontinuity separates D into two domains D_l and D_r (left and right) and the curve is written as

$$C(t) = \begin{pmatrix} t \\ x(t) \end{pmatrix}.$$

[1] William John Macquorn Rankine (5.7.1820-24.12.1872) war schottischer Mathematiker und Naturforscher, dessen Interessen zwischen Musik und Mathematik schwankten. Er beschäftige sich neben der Str"mungsmechanik mit vielen Anwendungen der Mathemaik.

[2] Pierre-Henri Hugoniot (5.6.1851-1887)

The unit tangent vector and unit normal vectors are

$$\dot{C}(t) = \frac{1}{\sqrt{1+\dot{x}^2}} \begin{pmatrix} 1 \\ \dot{x} \end{pmatrix}, \quad n(t) = \frac{1}{\sqrt{1+\dot{x}^2}} \begin{pmatrix} -\dot{x} \\ 1 \end{pmatrix}.$$

With this our integral can be split into left and right domains, the integration by parts will only produce boundary terms along the curve of discontinuities. We obtain

$$\begin{aligned} 0 &= \int\limits_D (u_t + (f(u))_x)\, \varphi \; d(x,t) \\ &= \int\limits_{D_l \cup D_r} (u_t + (f(u))_x)\, \varphi \; d(x,t) \\ &= \int\limits_{D_l} (u_t + (f(u))_x)\, \varphi \; d(x,t) + \int\limits_{D_r} (u_t - (f(u))_x)\, \varphi \; d(x,t). \end{aligned}$$

We concentrate on one of these two terms, without loss of generality on the first one, the second one is treated the same way, just not that at each point the outward normal unit vector is just the opposite of the one for D_l. Then we have

$$\begin{aligned} \int\limits_{D_l} (u_t + (f(u))_x)\, \varphi \; d(x,t) &= \int\limits_{D_l} u_t \varphi \; d(x,t) + \int\limits_{D_l} (f(u))_x \varphi \; d(x,t) \\ &= -\int\limits_{D_l} u \varphi_t \; d(x,t) + \int\limits_{\partial D_l} u \varphi d\omega - \int\limits_{D_l} (f(u)) \varphi_x \; d(x,t) + \int\limits_{\partial D_l} f(u) \varphi d\omega. \end{aligned}$$

Using the parametrization of C, the facts that unit normals for D_l and D_r are of opposite sign and that the limits for u on both sides of the boundary exist we find

$$0 = \int\limits_a^b \left((u_l - u_r)\dot{x} - (f(u_l) - f(u_r))\right) \varphi \frac{1}{\sqrt{1+\dot{x}^2}} ds.$$

Since this integral is zero for all C^1 functions on D and hence for all C^1-functions of the type $\varphi(x(t),t)$ on $[a,b]$ we conclude that the integrand has to be zero, ie

$$\dot{x}(u_l - u_r) = f(u_l) - f(u_r).$$

□

The speed $\dot{x}$ is precisely the shock speed and it is denoted by s. If we write $[\cdot]$ for the jump across the discontinuity we get the well known way to present this formula
$$s[u] = [f(u)].$$

2.2 Entropy solution

We observe that weak solutions are not unique. We show this by looking at an example, again we use the Burgers equation

$$u_t + uu_x = 0$$

with initial values
$$u_0(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}.$$

For fixed $\alpha > 0$ we look at the solution

$$u_\alpha(x,t) = \begin{cases} -1, & \text{for} \quad x < (-1-\alpha)\frac{t}{2} \\ -\alpha, & \text{for} \quad 0 > x > (-1-\alpha)\frac{t}{2} \\ \alpha, & \text{for} \quad 0 < x < (1+\alpha)\frac{t}{2} \\ 1, & \text{for} \quad x > (1+\alpha)\frac{t}{2} \end{cases}.$$

we can easily check that this function satisfies the Rankine-Hugoniot conditi-on and hence it is a solution of Burgers equation. This shows that the initial value problem has no unique solution in the class of weak solutions.
In a physical system there has to be a mechanism to select the physical meaningful solutions out the many possible solutions. The so called entropy condition gives such a mechanism.

Definition 2.2.1
A weak solution of a conservation law is called entropy solution *if there is a constant $E > 0$, such that for all $a > 0, t > 0$ and $x \in \mathbb{R}$ the so called* entropy condition *is satisfied*

$$\frac{u(x+a,t) - u(x,t)}{a} \leq \frac{E}{t}.$$

We will see shortly, that under certain hypotheses conservation laws possess a unique entropy solution. The entropy condition has an immediate consequence; it allows only downward jumps for increasing x. So none of the solutions u_α which we have given above satisfies the entropy condition. We find the entropy solution in the following form:

$$u_e(x,t) = \begin{cases} -1 & \text{for} \quad x < -t \\ \dfrac{x}{t} & \text{for} \quad -t < x < 1 \\ 1 & \text{for} \quad x > t \end{cases} .$$

Aufgabe 2.2.2

1. *Check that this is a solution of Burgers equations and it satisfies the entropy condition.*

2. *Prove that the entropy condition implies the entropy inequality: at a discontinuity we have the following inequality for the shock speed s:*

$$f'(u_l) > s > f'(u_r).$$

We come back to the question of existence of (entropy) solutions of conservation laws.

Satz 2.2.3
We consider the scalar conservation law

$$u_t + f(u)_x = 0$$

with $f \in C^2(\mathbb{R},\mathbb{R})$. Let $u_0 \in L^\infty(\mathbb{R})$ be given and assume $f'' > 0$ on $\left\{ u \in \mathbb{R} \;\middle|\; |u| \leq \|u_0\|_{L^\infty(\mathbb{R})} \right\}$. Then there exists a weak solution $u(x,t)$ of $u_t + (f(u))_x = 0$ and $u(x,0) = u_0(x)$ with the following properties.

1. $\|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$.

2. *There exists a constant $E > 0$, with $E = E(M,\mu,A)$, where $M = \|u\|_{L^\infty(\mathbb{R})}$, $\mu = \min\left\{ f''(u) \;\middle|\; |u| \leq M \right\}$, $A = \max\left\{ f'(u) \;\middle|\; |u| \leq M \right\}$ such that for $a > 0$, $t > 0$ and $x \in \mathbb{R}$ we have*

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}.$$

3. *u depends in the following sense continuously on initial condition: if $v_0 \in L^\infty(\mathbb{R})$ with $\|v_0\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}$ and if v denotes the solution corresponding to v_0 according to the first statement in this theorem, then for each pair $x_1 < x_2$ of real numbers and for every $t > 0$ we have:*

$$\int\limits_{x_1}^{x_2} |u(x,t) - v(x,t)| dx \le \int\limits_{x_1 - At}^{x_2 + At} |u_0(x) - v_0(x)| dx. \qquad (2.6)$$

For a proof consult Smoller [18].

Satz 2.2.4
If $f \in C^2(\mathbb{R})$ and $f'' > 0$ and if u, v are two weak solutions of the initial value problem which satisfy the entropy condition then we have $u = v$ almost everywhere.

Again we refer to [18] for a proof.

2.3 The Riemann Problem

Let us consider the equation

$$u_t + Au_x = 0$$

for $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ with a $n \times n$-matrix A. We make an Ansatz

$$u(x,t) = \xi \exp(i(\lambda t + \mu x)).$$

If we put this into the equation we obtain the condition

$$\mu A \xi = -\lambda \xi$$

or ξ is an eigenvector for the eigenvalue $\frac{\lambda}{\mu}$ $\mu \neq 0$. If we require, that the initial value

$$u(x,0) = \xi \exp(i\mu x)$$

is bounded we obtain immediately that μ is real. Our next requirement is more subtle. We want that small initial data does not lead to large solutions, this means (by a simple computation) that $\frac{\lambda}{\mu}$ is real.

Definition 2.3.1
If all eigenvalues of A are real we call equation

$$u_t = Au_x$$

weakly hyperbolic. The equation is called hyperbolic if in addition all eigenvalues of A are distinct.

Now we want to derive an entropy inequality for such systems. For pedagogical reasons we start with a system

$$u_t + au_x = 0$$

on $\mathbb{R}^+ \times \mathbb{R}^+$. From our previous considerations it is clear that u is constant on lines $x - at = c$. If $a < 0$ these lines intersect both the initial time $t = 0$ as the boundary of the domain $x = 0$. Therefore describing initial values determines the solution completely and it is not possible to prescribe boundary values as well. On the other hand for $a > 0$ the initial values determine the solution just on a part of the quarter plane and in order to have solution on the whole quarter plane we have to prescribe boundary values. If we go the case of a hyperbolic system, we get eigenvalues

$$\lambda_1 < \cdots < \lambda_k < 0 < \lambda_{k+1} < \cdots < \lambda_n.$$

Diagonalizing A means to find a matrix P with $P^{-1}AP = \Lambda = \text{diag } (\lambda_1, \ldots, \lambda_n)$ leads to the equation

$$v_t = \Lambda v_x$$

with $v = P^{-1}u$. This is a set of uncoupled equations on the quarter plane. Now we have k equations with $\lambda_i < 0$ and $n - k$ with the corresponding eigenvalue positive. So we have to specify $n - k$ boundary conditions $v^i(0, t)$ in order to get solutions on the whole quarter plane. Going back into the equation for u we observe that we have to specify $n - k$ boundary conditions for u in order to get a unique solution.

Beispiel 2.3.2
Assume u, v are elements in $\mathbb{R}^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then we have

$$\Lambda = diag\ (1, -1_1, \ldots, 1, -1_n)\,.$$

We have to specify v_1 at $x = 0$ to get a unique solution for the v-system. In the u-coordinates we obtain the condition

$$u_1(0,t) = -u_2(0,t).$$

This can be directly read off from the matrix P.

If we replace the boundary at $x = 0$ by a line $x = st$ and if we have eigenvalues of A of the form

$$\lambda_1 < \cdots < \lambda_k < s < \lambda_{k+1} < \cdots < \lambda_n$$

then we have, by the same reasoning, to specify $n - k$ boundary conditions along this line.

Now we want to come back to the shock conditions. It is obvious that for the linear systems the eigenvalues of df are the same on both sides of the shock and so we get the same conditions on both sides. The Rankine-Hugoniot condition implies immediately that shock lies are eigenspaces of A and hence we cannot satisfy the condition that the shock speed is between eigenvalues. Therefore we look at the nonlinear problem, we have $P = P(u)$, $\Lambda = \Lambda(u)$. We assume that $df(u)$ has eigenvalues

$$\lambda_1(u) < \cdots < \lambda_k(u) < \cdots < \lambda_n(u).$$

Moreover let u_l, u_r denote the left and the right value along the discontinuity. Assume

$$\lambda_k(u_r) < s < \lambda_{k+1}(u_r).$$

Then following the previous argument we should specify $n - k$ values o the right side of the discontinuity which is given by $P(u_r)$. On the left side we assume

$$\lambda_j(u_l) < s < \lambda_{j+1}(u_l).$$

the same argument as before indicates to specify j conditions on the left side. The Rankine-Hugoniot condition gives n relations between the values on the left and on the right. We use of these equations to eliminate s, the shock speed from the problem. Therefore we have $n - k + j$ conditions to get a unique solution.

$$n - k + j = n - 1$$

yielding

$$j = k - 1.$$

Therefore we allow for discontinuities at $x = st$ for

$$\lambda_k(u_r) < s < \lambda_{k+1}(u_r)$$

and

$$\lambda_{k-1}(u_l) < s < \lambda_k(u_l).$$

If we have a discontinuity with satisfying these conditions we call a k-shock and the inequalities which we just discussed are called *entropy inequalities* or *Lax shock conditions*[3]

[3]Peter Lax (1.5.1926–) is a contemporary American mathematician working at the Courant Institute in New York. He has significantly contributed to the theory of partial differential equations. His major achievements concern hyperbolic conservation laws and some aspects of elliptic equations. He received the Abel prize in 2005.