Sectorial operators generate analytic semigroups

Reiner Lauterbach

Faculty for Mathematics, Informatics und Sciences
Universität Hamburg

Ergänzungen

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Operators of type (Φ, M)

Definition. Let $A : X \to X$ be a linear operator with $D(A) \subset X$. It is called an operator of type (Φ, M) for $\Phi \in (0, \frac{\pi}{2})$, $M > 0$ if

1. A is closed and $D(A)$ is dense in X.
2. The resolvent set $P(A)$ of A contains the set

$$S_\Phi = \left\{ z \in \mathbb{C} \mid z \neq 0,\ \frac{1}{2}\pi - \Phi < \arg(z) < \frac{3}{2}\pi + \Phi \right\}$$

   and for all $\lambda \in S_\Phi$ we have an estimate of the resolvent as

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}. \tag{1}$$

The operator A is called sectorial, if there exists a $\tau \in \mathbb{R}$ with $A - \tau \mathbb{1}$ is of type (Φ, M).

Statement of the Theorem

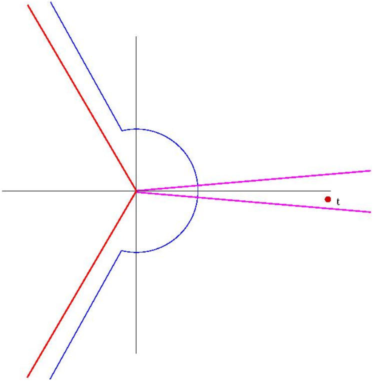
If A is sectorial, then $-A$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$. Moreover we have:

1. For $t > 0$ the operators $AT(t)$, $\frac{d}{dt}T(t)$ are bounded linear with
$$\frac{d}{dt}T(t)x = -AT(t)x, \;\; \forall x \in X, \;\; t > 0.$$
2. T has an analytic continuation to a sector
$$S = \left\{ z \in \mathbb{C} \;\middle|\; |\arg z| < \varphi_1 \right\}$$
which contains the positive real half axis. For $t, \; s \in S$ we have $T(t+s) = T(t) \circ T(s)$.
3. There exists an $a \in \mathbb{R}$, such that for all $t \in S$ we have an estimate of the following form
$$\|T(t)\| \leq Ce^{-at} \text{ and } \|AT(t)\| \leq \frac{C}{|t|}e^{-at}.$$

Proof – Preliminiaries

Wlog: $\tau = 0$.

Curve Γ



The left (red) curve indicates the boundary of the set which by assumption contains the resolvent set of $-A$. The blue curve represents Γ. If we multiply t with $\lambda \in \Gamma$, then $\mathrm{Re}(\lambda t) < 0$ and $\lim_{|\lambda|\to\infty} \mathrm{Re}(\lambda t) = 0$.

Proof – Construction of $T(t)$

We define

$$T(t)x = \frac{1}{2\pi i} \int\limits_{\Gamma} R(\lambda, -A) \exp(\lambda t) x \, d\lambda.$$

We write $\Gamma_n = \Gamma \cup B_n(0)$ and $\Gamma_{c,n} = \Gamma \setminus \Gamma_n$. Then

$$T(t)x = \frac{1}{2\pi i} \int\limits_{\Gamma_n} R(\lambda, -A) \exp(\lambda t) x \, d\lambda + \frac{1}{2\pi i} \int\limits_{\Gamma_{c,n}} R(\lambda, -A) \exp(\lambda t) x \, d\lambda.$$

We estimate

$$\left\| \frac{1}{2\pi i} \int\limits_{\Gamma_{c,n}} R(\lambda, -A) \exp(\lambda t) x \, d\lambda \right\| \leq \frac{1}{2\pi} \int\limits_{\Gamma_{c,n}} \frac{M}{|\lambda|} \exp(Re(\lambda t)) \, d\lambda \leq \frac{MC(n)}{2\pi n}$$

Proof – Convergence of $T(t)$

Therefore

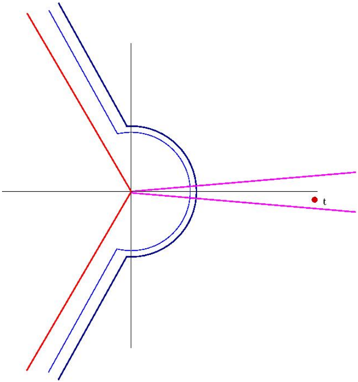
$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\Gamma_n} R(\lambda, -A) \exp(\lambda t) x \, d\lambda$$

exists and

$$T(t)x = \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, -A) \exp(\lambda t) x \, d\lambda.$$

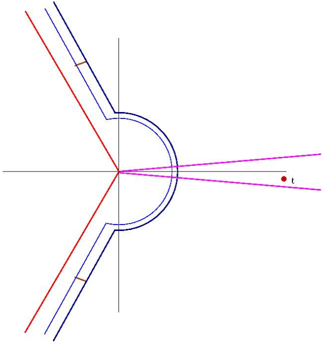
is defined.

Proof – Semigroup Property – Curve Γ_0



The dark blue curve indicates a new curve Γ_0. To show that using Γ_0 we get the same integral, we choose $f \in X^*$ and $x \in X$. Define curves $\Gamma_{0,n}$ and $\Gamma_{0,c,n}$ similar to the above construction.

Proof – Semigroup Property – Independence of curve



We integrate

$$g(\lambda, t) = f\left(R(\lambda, -A)\exp(\lambda t)x\right)$$

along Γ and Γ_0 and take the difference. Denote the brown arcs by A_1 and A_2, such that $\Gamma_n + A_1 - \Gamma_{0,n} - A_2$ is a closed curve γ.

$$\left| \int_\Gamma g(\lambda, t)\, d\lambda - \int_{\Gamma_0} g(\lambda, t)\, d\lambda \right| \leq \left| \int_\gamma g(\lambda, t)\, d\lambda \right| + \\ + \int_{|\Gamma_{c,n}| + |\Gamma_{0,c,n}| + |A_1| + |A_2|} g(\lambda, t)\, d\lambda$$

Proof – Semigroup Property – Cauchy and Estimates

By Cauchy's theorem we have

$$\int_{\gamma} g(\lambda, t)\, d\lambda = 0.$$

Then for $N = \Gamma_{c,n}$ or $N = \Gamma_{0,c,n}$ we have

$$\left| \int_{N} g(\lambda, t)\, d\lambda \right| \leq \frac{M}{n} \|f\|_{X^*} \int_{N} \exp(\mathrm{Re}\, \lambda t) \mathrm{d}\lambda \to 0 \text{ as } n \to \infty.$$

For A_i

$$\left| \int_{A_i} g(\lambda, t)\, d\lambda \right| \leq \frac{M}{n} \|f\|_{X^*} \int_{A_i} \exp(\mathrm{Re}\, \lambda t) \mathrm{d}\lambda \to 0 \text{ as } n \to \infty.$$

Proof – 1.) Growth condition

We use the estimate from above for $R(\frac{\lambda}{|t|}, -A)$, i.e.

$$\|R(\frac{\lambda}{|t|}, -A)\| \leq \frac{M|t|}{|\lambda|}$$

to obtain with $\Gamma' = |t|\Gamma$, a change of coordinates $\lambda' = \lambda|t|$ and $t = |t|\xi$, $|\xi| = 1$ and Cauchy's Theorem (replacing Γ' by Γ):

$$\|T(t)\| \leq \frac{M}{2\pi} \int_{\Gamma} |\exp(\lambda'\xi)| \frac{d\lambda'}{\lambda'} = \frac{MC(\xi)}{2\pi}.$$

Proof – 2.) Growth condition

Since A is closed, we can write A into the integral over Γ (we had such an argument several times before). Therefore we have an expression for $AT(t)$ given by

$$AT(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda' \xi) AR(\frac{\lambda'}{|t|}, -A) \frac{d\lambda'}{|t|}.$$

We get

$$AT(t) = \frac{1}{2\pi i} \int_{\Gamma} \exp(\lambda' \xi)(-\mathbb{1} + \frac{\lambda'}{|t|}) R(\frac{\lambda'}{|t|}) \frac{d\lambda'}{|t|}.$$

The term with the $\mathbb{1}$ gives a constant, the second term yields an esitmate of the form

$$\frac{M}{2\pi} \int_{\Gamma} |\exp(\lambda' t)| \frac{\lambda'}{|t|} \frac{|t|}{\lambda'} \frac{d\lambda'}{|t|}.$$

This yields an estimate of the form $\leq \frac{C}{|t|}$.

Proof – Strong Continuity

1. For $x \in D(A)$ we have

$$T(t)x - x = \frac{1}{2\pi i}\int_\Gamma \exp(\lambda t)[R(\lambda, -A) - \lambda^{-1}]x\, d\lambda$$

$$= -\frac{1}{2\pi i}\int_\Gamma \lambda^{-1} e^{\lambda t} AR(\lambda, -A)x\, d\lambda.$$

This gives for $t > 0$

$$\|T(t)x - x\| \leq C\|Ax\|t$$

(use $\mu = \lambda t$.) (Observe $R(\lambda, -A)(\lambda \mathbb{1} + A) = \mathbb{1}$ and hence $AR(\lambda, -A)\lambda^{-1} + R(\lambda, -A) = \lambda^{-1}\mathbb{1}$ implying $R(\lambda, -A) - \lambda^{-1} = -AR(\lambda, -A)\lambda^{-1}$)

2. $D(A) \subset X$ is dense $T(t)$ bounded:
$\|T(t)x - x\| \leq \|T(t)x - T(t)x_n\| + \|T(t)x_n - x_n\| + \|x_n - x\|$.

Proof – Generator of the Semigroup – $-A \subset \bar{A}$

Let $x \in D(A)$and $t > 0$, then

$$\frac{d}{dt}AT(t) + AT(t) = \frac{1}{2\pi i}\int\limits_{\Gamma} e^{\lambda t}(\lambda + A)R(\lambda, -A)x \, d\lambda = 0.$$

Then for $x \in D(A), \; t > 0$ we have

$$\frac{1}{t}\int\limits_0^t \frac{d}{ds}T(s)xds = -\frac{1}{t}\int\limits_0^t T(s)Ax \, ds.$$

The limit on the right hand side exists and yields $-Ax$. Therefore $-A$ is containted in the generator $\bar{A}$

Proof – Generator of the Semigroup – $\bar{A} = -A$

Define for λ sufficiently large (given by the growth rate ω of $T(t)$)

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

$R = R(\lambda)$ is bounded linear operator.

Consider for $x \in X$

$$\begin{aligned}
\frac{1}{h}(T(h)Rx - Rx) &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t} T(t+h)x\,dt - \int_0^\infty e^{-\lambda t} T(t)x\,dt\right) \\
&= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda u} T(u)x\,du - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x\,dt \\
&= \frac{e^{\lambda h}-1}{h}\int_h^\infty e^{-\lambda u} T(u)x\,du - \frac{1}{h}\int_0^h e^{\lambda t} T(t)x\,dt \\
&\to \lambda Rx - x \text{ for } h \to 0.
\end{aligned}$$

This implies for $x \in X$ that $R(\lambda)x \in D(\bar{A})$ and $\bar{A}Rx - \lambda Rx = -x$ i.e. we have $(\lambda \mathbb{1} - \bar{A})R(\lambda)x = x$ for all $x \in X$. Then, trivially, for $x \in D(\bar{A})$ we have $R(\lambda)x \in D(\bar{A})$ and

$$\bar{A}R(\lambda)x = \bar{A}\int_0^\infty e^{\lambda t}T(t)x\,dt = \int_0^\infty e^{\lambda t}T(t)\bar{A}x\,dt = R(\lambda)\bar{A}x.$$

Therefore we have $\bar{R}Ax - \lambda Rx = -x$ and hence

$$R(\lambda)(\lambda \mathbb{1} - \bar{A})x = x \text{ for } x \in D(\bar{A}).$$

Therefore $R(\lambda) = (\lambda \mathbb{1} - \bar{A})^{-1}$, Then $P(A) \cap P(\bar{A}) \neq \emptyset$. For $\lambda \in D(A) \cap D(\bar{A})$ we have

$$(\lambda \mathbb{1} - \bar{A})D(A) = (\lambda \mathbb{1} - A)D(A) = X.$$

Since $\lambda \mathbb{1} - \bar{A}$ is injective and $D(A) \subset D\bar{A})$ we have

$$D(A) = D(\bar{A}).$$