Geometric topology

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Summary of topics

1. ## Differential topology basics

   (a) basic definitions: (smooth) manifolds, submersions, immersions, embeddings, regular and critical points and values
   (b) different versions of Whitney's embedding theorem
   (c) statement of Sard's theorem, applications
   (d) transversality: definition, consequences, basic examples
   (e) transversality theorem for families and consequences, in particular how to make maps transverse to a given map
   (f) mapping degree (mod 2 and integer valued), linking number for disjoint closed curves in $\mathbb{R}^3$
   (g) vector bundles: definition and basic constructions
   (h) normal bundle and tubular neighborhood theorem
   (i) orientations of transverse intersections and intersection numbers
   (j) index of an isolated zero of a vector field, Euler characteristic and Poincaré-Hopf theorem
   (k) maps into spheres are classified by degree
   (l) Whitney's trick and the strong Whitney embedding theorem

2. ## Gluing constructions for manifolds

   (a) uniqueness of disk embeddings up to isotopy
   (b) connected sum of manifolds
   (c) gluing of manifolds along embedded copies of a submanifold with matching normal bundles: construction and examples
   (d) special case: surgery
   (e) handle attachment, relation to surgery, boundary connected sum as special case

3. ## Elements of Morse theory

   (a) basic definitions: nondegenerate critical point, index, Morse function, stable and unstable manifolds
   (b) Morse lemma
   (c) topology of sublevels changes only at critical levels
   (d) description of the change via handle attachment
   (e) Morse-Smale property and how to achieve it
   (f) existence of self-indexing Morse functions
   (g) construction of the Morse complex, computations in examples

4. ## Topology of three-manifolds

   (a) examples
   (b) Heegaard splittings and Heegaard diagrams: existence and examples
   (c) Dehn twists and the Dehn-Lickorish theorem (including sketch of proof)
   (d) Dehn surgery, examples
   (e) surgery theorem (including sketch of proof)

Advice for your exam preparation

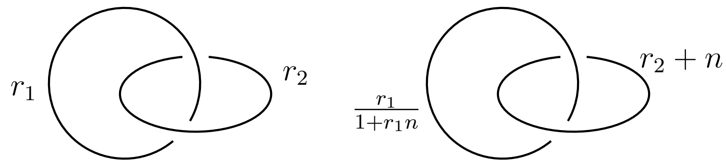
Of course you will need to know the statements and proofs of the main results. Make sure you also know and understand *many* examples. I sometimes ask what happens to a given theorem when you leave out one of the assumptions. Do you know counterexamples? What is a simple situation where a given theorem is useful?

As you prepare for the exam, look back at the exercises, as they often give a valuable second perspective on topics covered in the lecture. Also, you might find it useful to look up the treatment of the topics we covered in the textbooks by Hirsch and Kosinski (for the differential topology and surgery parts), Audin/Damian (for the Morse theory part) and Sossinsky/Prasolov and Rolfsen (for the part on 3-manifolds).

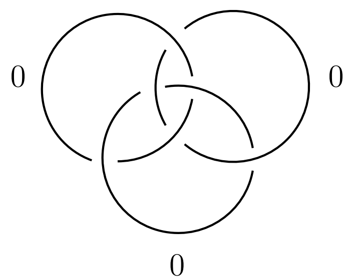
The duration of the exam is between 25 and 30 minutes.
You may choose to take it in either in late February (Thursday, Feb 22 or Tuesday, Feb 27), or after March 25 (either in that week, or in the first week of April).
Please let me know by mail before February 10 if you want to take the exam in February.

You may choose to start the exam by presenting your solution to one of the following problems:

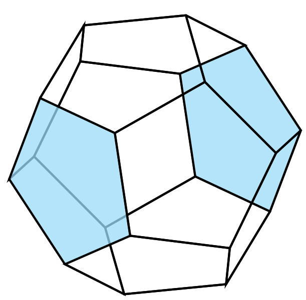
- Prove that for integers r_1, r_2 and n the following two surgery diagrams in S^3 always produce the same 3-manifold, and identify that manifold for $r_1 = r_2 = 1$.



- Prove that the result of surgery on the Borromean link in S^3 with surgery coefficients 0 on all three components is T^3.
  Hint: Find a suitable Heegaard surface of genus 3 whose Heegaard diagram is equivalent to the one we discussed for T^3 in the exercise class.



- Explain the method of Seifert and Threlfall[1] to obtain a Heegaard diagram of genus 2 for the manifold obtained by gluing opposite faces of a dodecahedron by $\frac{1}{10}$ of a full twist.



- Find a presentation of the fundamental group of the 3-manifold obtained by gluing opposite faces of a dodecahedron by $\frac{3}{10}$ of a full twist, and show that the group is not finite.

[1]§12 in W. Threlfall, H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Annalen 104, 1931, 1–70, `https://doi.org/10.1007/BF01457920`.