Universität Hamburg
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Geometric Topology

Problem Set 9

1. In class, we proved that any finitely presented group arises as fundamental group of a closed oriented 4-manifold. Describe as precisely as possible what goes wrong when one tries to produce a closed oriented 3-manifold with a given fundamental group by an analogous construction.

2. Let M be a closed oriented 3-manifold, and let $f : M \to \mathbb{R}$ be a self-indexing Morse function on M.
Prove that the subsets $M^{\leq \frac{3}{2}}$ and $M^{\geq \frac{3}{2}}$ are two handlebodies with the same number of handles. Here, the handlebody with k handles is defined as the boundary connected sum of k copies of $S^1 \times D^2$.

3. **a)** Use the discussion of Morse-Smale gradient flows and the 'rearrangement lemma' proven in class to show that any Morse function on a closed manifold can be modified into a self-indexing Morse function with precisely the same critical points, having precisely the same indices.

b) Alternatively, prove the existence of self-indexing Morse functions by assuming the fact that any smooth closed manifold admits a smooth triangulation. This method typically produces functions with many more critical points than necessary.

4. Let M be a connected compact manifold with boundary and let $f : M \to [a, b]$ be a Morse function with the following properties:

- a and b are regular values of f, and $\partial M = f^{-1}(a) \cup f^{-1}(b)$ (a priori one or both of $f^{-1}(a)$ and $f^{-1}(b)$ could be empty)
- f has exactly three critical points, two of index 0 and one of index 1.

a) Prove that in fact $f^{-1}(a)$ must be empty, and M must be diffeomorphic to a closed ball whose boundary is a regular level set. In particular, there is a Morse function $g : M \to [a, b]$ with a single local minimum such that $g \equiv f$ near ∂M.

b) Deduce that every closed connected manifold M admits a Morse function with a single local minimum and a single local maximum.