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Geometric Topology

Problem Set 6

1. Suppose V is a vector field on $\mathbb{R}^n$ whose restriction to S^{n-1} is nowhere tangent to S^{n-1}. Prove that V must have a zero at some point in the ball $B^n \subseteq \mathbb{R}^n$.

2. **a)** Describe a vector field on S^2 with a single zero.

b) Find a function $f : T^2 \to \mathbb{R}$ with exactly three critical points. What are the indices of its gradient vector field (with respect to any metric) at the three critical points?

3. **a)** Let M and N be compact manifolds with $\partial N = 0$. Prove that $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

b) Prove that if $p : E \to M$ is a vector bundle over a closed manifold and $D \subseteq E$ is a disk subbundle (i.e. $D \subseteq E$ is a compact submanifold with boundary which intersects each fiber E_x in a disk around $0 \in E_x$), then $\chi(D) = \chi(M)$.

c) Suppose that M and N are compact manifolds (possibly with boundary) and $f : M \to N$ is a d-sheeted covering map. Prove that $\chi(M) = d \cdot \chi(N)$.

d) What is $\chi(\mathbb{R}P^n)$?

4. **a)** Suppose a closed manifold M can be written as $M = A \cup B$, where $A, B \subseteq M$ are compact subdomains of the same dimension with common boundary $\partial A = \partial B = A \cap B$. Prove that $\chi(M) = \chi(A) + \chi(B) - \chi(A \cap B)$.

b) Use this to give an alternative computation of $\chi(\mathbb{R}P^n)$.

c) Prove that if a closed manifold C is the boundary of some compact manifold, then $\chi(C)$ is even.

d) Prove that for even n the manifolds $\mathbb{R}P^n$ and $\mathbb{C}P^n$ cannot be boundaries of compact manifolds.

e) Find a compact 3-dimensional manifold whose boundary is a Klein bottle.