Universität Hamburg
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Geometric Topology

Problem Set 2

1. Let M be a connected manifold.

a) Prove that for any two points $p, q \in M$, there exists a smooth *embedded* path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.
How many substantially different proofs can you find?

b) Prove that for any two points $p, q \in M$ there is a diffeomorphism $\varphi : M \to M$ with $\varphi(p) = q$.
Moreover, the diffeomorphism φ can be chosen as the end point of a smooth path of diffeomorphisms starting at id_M. Such a path is called an isotopy.

2. Let M be a closed manifold. Prove that there exist finitely many vector fields $X_1, \ldots, X_N$ with the property that at each $p \in M$ the vectors $X_1(p), \ldots, X_N(p)$ span T_pM. In other words, the map of vector bundles

$$\begin{aligned} \Phi : M \times \mathbb{R}^N &\to TM \\ (p, t_1, \ldots, t_N) &\mapsto \sum_{r=1}^{N} t_r X_r(p) \end{aligned}$$

is surjective.

3. Let M be a compact manifold with boundary. Prove that every neighborhood $W \subseteq M$ of ∂M contains a neighborhood $U \subseteq W$ of ∂M which is diffeomorphic to $\partial M \times [0,1)$ by a diffeomorphism sending $(p,0)$ to p. Such a U is called a *collar neighborhood of ∂M*.
Hint: This is true without assuming M is compact, but the proof is slightly more technical.