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Winter 2023/24

# Geometric Topology

## Problem Set 11

**1.** Prove that under the embedding $\mathbb{C}^2 \to \mathrm{Mat}(2,\mathbb{C})$ given by the mapping

$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

**a)** $S^3$ is diffeomorphic to the Lie group $SU(2) \subseteq \mathrm{Mat}(2,\mathbb{C})$,

**b)** the action of $SU(2) \times SU(2)$ on $\mathbb{C}^2 \cong \mathbb{R}^4$ by left and right multiplication gives rise to a surjective group homomorphism

$$SU(2) \times SU(2) \to SO(4)$$

with kernel $\{(\mathbb{1}, \mathbb{1}), (-\mathbb{1}, -\mathbb{1})\}$, and

**c)** the corresponding action of $SU(2) \times SU(2)$ on $S^3 \subset \mathbb{R}^4$ descends to a faithful action of $SO(3) \times SO(3) \cong SU(2)/\pm\mathbb{1} \times SU(2)/\pm\mathbb{1}$ on $\mathbb{R}P^3 = S^3/\pm\mathbb{1}$.

**2.** **a)** Can you find two different Heegaard diagrams of genus 2 for $S^3$?

**b)** Find a Heegaard diagram of genus 3 for $T^3$.

**c)** Can you describe a procedure to produce a Heegaard diagram of genus $g+1$ for a given 3-manifold $M$ from a Heegaard of genus $g$ for the same $M$?

**3.** Let $\alpha$ and $\beta$ denote loops representing the standard generators of $\pi_1(T^2 \setminus \{pt\})$, and let $\tau_\alpha$ and $\tau_\beta$ be the corresponding Dehn twists on $T^2 \setminus \{pt\}$. Prove that

**a)** $\tau_\beta \tau_\alpha \tau_\beta(\alpha)$ is isotopic to $\beta$ and

**b)** $\tau_\beta \tau_\alpha \tau_\beta(\beta)$ is isotopic to $\alpha^{-1}$ (meaning $\alpha$ traversed in the opposite direction).