Universität Hamburg
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Geometric Topology

Problem Set 10

1. Compute the Morse homology with coefficients in $\mathbb{Z}_2$ of the Morse functions on $\mathbb{R}P^n$ and on $\mathbb{C}P^n$ discussed in Problem 1 of Problem set 8.

2. The goal of this exercise is to classify closed connected oriented surfaces. Recall that a 3-dimensional handle body with g handles is the boundary connected sum of $g \geq 0$ copies of $S^1 \times D^2$. Two such handle bodies are diffeomorphic if and only if they have the same number of handles. We call the boundary of a handle body with g handles an *oriented surface of genus g*. Our goal is to show that every closed connected oriented surface is diffeomorphic to a unique oriented surface of genus g for some $g \geq 0$.

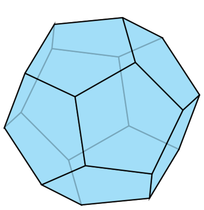
a) Compute the Euler characteristic of an oriented surfaces of genus g and conclude that two such surfaces of different genus are not diffeomorphic.

b) Suppose that Σ is a connected oriented surface with boundary and $f : \Sigma \to \mathbb{R}$ is a Morse function with a single critical point of index 1 and such that $\partial\Sigma$ is a union of regular level sets. Prove that Σ is diffeomorphic to a pair of pants, i.e. a sphere with three open disks (with disjoint closures) removed.

c) Now let Σ be a closed, connected oriented surface and $f : \Sigma \to \mathbb{R}$ a Morse function. Assume that the critical values are distinct for distinct critical points, and let $p \in \Sigma$ be the index 1 critical point with the lowest value $f(p)$. Classify the topological types that the component $\Sigma_0 \subseteq \Sigma^{\leq f(p)+\varepsilon}$ containing p can have. *Hint: There are only two options.*

d) Now argue by induction on the number n of critical points of index 1 of a Morse function $f : \Sigma \to \mathbb{R}$ that every closed connected oriented surface is diffeomorphic to a surface of some genus $0 \leq g \leq n$. *Hint: Problem 4a) of Problem Set 9 might be useful here.*

3. A dodecahedron is a regular 3-dimensional polytope with regular pentagons as faces. Two pentagons meet at each edge, and three pentagons meet at each vertex.



a) How many faces, edges and vertices does the dodecahedron have?

b) Opposite faces of the dodecahedron do not quite align, but we can glue them with a twist of $\frac{1}{10} \cdot 2\pi$ in (say) the clockwise direction as we go from front to back. If we perform this gluing with all pairs of opposite pentagons, then how many vertices and edges does the resulting cell complex M_1 have? Is it a (topological) 3-manifold?

c) Repeat the analysis in (b) for the space M_2 obtained by gluing opposite pentagons with a twist of $\frac{3}{10} \cdot 2\pi$.

d) Are the two resulting spaces homeomorphic?

e*) How many different *oriented* 3-manifolds can you obtain from gluing opposite faces of a cube? How can you decide which of your gluings produce different manifolds?