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Differential Topology

Problem Set 2

1. Prove that any topological manifold of dimension $n \in \mathbb{N}$ according to our definition (Hausdorff, second countable, locally homeomorphic to $\mathbb{R}^n$) is *paracompact*, meaning that every open covering has a locally finite refinement.
Recall: A refinement *of an open covering* $\{U_\alpha\}_{\alpha \in A}$ *is an open covering* $\{W_\beta\}_{\beta \in B}$ *such that for each* $\beta \in B$ *there exists some* $\alpha \in A$ *with* $W_\beta \subseteq U_\alpha$.
An open covering is locally finite *if every point has a neighborhood meeting only finitely many of the sets of the covering.*

2. Give a proof of the Whitney embedding theorem for noncompact manifolds: Every smooth manifold of dimension n has an embedding as a submanifold into $\mathbb{R}^{2n+1}$ such that the image is a closed subset.
Hint: You may find the following result useful: Every open cover of a topological manifold M *admits a locally finite refinement* $\{V_\alpha\}_{\alpha \in A}$ *such that any point* $x \in M$ *meets at most* $\dim M + 1$ *of the open sets* V_α. *(You do not need to prove this, although it might be fun trying.)*
Possible strategy: Assuming the original cover to be by charts with bounded image in $\mathbb{R}^n$, *the* V_α *will still be chart domains. Now partition the index set* α *into finitely many subsets such that the* V_α *for* α *in a given subset are disjoint, and use these to build an embedding into* $\mathbb{R}^d$ *for a suitable d, along the lines of the proof of Theorem 1. Use an arbitrary additional proper function (show existence!) to make the embedding closed, and then use the argument from the proof of Theorem 2 to conclude.*

3. Prove the following assertion from the lecture: If $U \subseteq \mathbb{R}^n$ is open and $A \subseteq U$ has measure zero and if $f : U \to V \subseteq \mathbb{R}^n$ is a map of class C^1, then $f(A)$ has measure zero in V.

4. Let $\{f_k\}_{k \geq 1}$ be a sequence of maps converging to a map f in $C^s(M, N)$ for some $s \geq 0$ with the strong topology. Show that there exists a compact subset $K \subseteq M$ and some $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $x \in M \setminus K$ one has

$$f_k(x) = f(x).$$

5. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets and let $0 \leq s < r$.

a) Prove that the subset $C^s(U, V) \subseteq C^s(U, \mathbb{R}^m)$ is open in the strong C^s topology. Is this true for the weak C^s topology?

b) Prove that the embedding $C^r(U, V) \subseteq C^s(U, V)$ is continuous.

We could spend some time on the first or the last exercise during the recitation.