Universität Hamburg
Janko Latschev

Winter 2013/14

Differential Topology

Problem Set 2

1. Each standard open set $U_j := \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n \mid z_j \neq 0\}$ is diffeomorphic to $\mathbb{R}^{2n}$, and its complement can be identified with $\mathbb{C}P^{n-1}$.

a) Prove that the intersection of U_j with a suitable open neighborhood V_j of $\mathbb{C}P^n \setminus U_j$ is homotopy equivalent to S^{2n-1}!

b) Use this to compute the de Rham cohomology of $\mathbb{C}P^n$ inductively using the Mayer-Vietoris sequence!

2. **a)** Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology of the complement of k disjoint small (closed) balls inside a bigger (open) ball in $\mathbb{R}^n$!

b) Use this result in the case $n = 2$ to compute the de Rham cohomology of a closed oriented surface of genus g, by writing it as a union of two disks with $g + 1$ holes as in **a)** (think of the "standard" drawing and take slight enlargements of the front half and the back half)!

3. **a)** Prove that every smooth map $S^2 \to T^2$ has degree zero!
Hint: One solution to this problem involves the ring structure on de Rham cohomology.

b) Prove the same result for maps $S^2 \to \Sigma_g$ to surfaces of higher genus $g \geq 2$!

4. What are the possible degrees of maps from T^2 to itself? Are homotopy classes of such maps classified by their degree?

5. The aim of this exercise is to prove the statement made in class that the top-dimensional de Rham cohomology group $H^n_{\mathrm{dR}}(M)$ of a closed, connected and oriented n-dimensional manifold is isomorphic to $\mathbb{R}$.

a) Prove by induction on n that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function with compact support and $\int_{\mathbb{R}^n} f(x)\, dx_1 \ldots dx_n = 0$, then there exist functions $u_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, n\}$ with compact support such that $f = \sum_i \frac{\partial u_i}{\partial x_i}$.
Hint: The case $n = 1$ is an easy consequence of the fundamental theorem of calculus. For the induction step consider the auxiliary function

$$g(x_2, \ldots, x_n) := \int_{\mathbb{R}} f(x_1, x_2, \ldots, x_n)\, dx_1,$$

and observe that by Fubini's theorem one can apply the induction hypothesis to obtain $u_2 \ldots, u_n$. To get the remaining function u_1, adjust

$$w_1(x_1, \ldots, x_n) := \int_{-\infty}^{x_1} f(t, x_2, \ldots, x_n)\, dt$$

by subtracting a suitably cut off version of g.

Bitte wenden!

b) Deduce from this that every compactly supported form $\omega \in \Omega^n(\mathbb{R}^n)$ with vanishing integral is the differential of a compactly supported form $\eta \in \Omega^{n-1}(\mathbb{R}^n)$.

c) Now prove that for a manifold M satisfying the above assumptions, there are finitely many open sets $U_0, U_1, \ldots, U_r$ diffeomorphic to balls and covering M and diffeomorphisms $\varphi_i : M \to M$ isotopic to the identity with $\varphi_i(U_0) = U_i$.

d) Prove that for any closed form $\alpha \in \Omega^n(M)$ with compact support in some U_i, $\varphi_i^*\alpha$ and α are cohomologous.
Hint: Consider an isotopy $\Phi_t : M \to M$, $t \in [0,1]$ with $\Phi_0 = id_M$ and $\Phi_1 = \varphi_i$. Now argue that for $t, t' \in [0,1]$ sufficiently close, $\Phi_t^*\alpha$ and $\Phi_{t'}^*\alpha$ will both have support in $\Phi_t^{-1}(U_i)$ and have the same integral, and so by part **b)** *they must be cohomologous. Finish with a standard open-and-closed argument.*

e) Now complete the proof of the original claim by using a partition of unity subordinate to the cover $\{U_i\}_{i=0,\ldots,r}$ of M from part **c)** to break up a given form $\omega \in \Omega^n(M)$ whose integral over M vanishes into components ω_i with support in U_i and applying the result of part **b)** to the form

$$\tilde{\omega} = \sum_{i=0}^{n} \varphi_i^* \omega_i$$

with support in U_0, which by part **d)** is cohomologous to ω.

6. Prove that for any smooth map $\varphi : M \to M$ of a manifold which is homotopic to the identity, the induced map $\varphi^* : H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{dR}}(M)$ is the identity.
Note that this substantially generalizes part **c)** *of the previous exercise.*

7. Observe that for any contracible open subset U of a manifold M, the set of orientations of the tangent spaces of the points $p \in U$ can be identified with $U \times \{\pm 1\}$, which has an obvious projection to U.

a) Use this observation to construct a covering space $\pi : \widehat{M} \to M$ whose fiber at a point $p \in M$ consists of the two possible orientations for T_pM and prove that this double covering $\widehat{M}$ is an orientable manifold. It is called the *oriented double covering of* M.

b) Prove that a connected manifold M is orientable if and only if $\widehat{M}$ is disconnected.

c) Prove that for the Klein bottle K^2, the oriented double covering $\widehat{K^2}$ is diffeomorphic to T^2.

8. Given a C^2-function $f : M \to \mathbb{R}$ and a critical point $p \in M$ (i.e. a point where $df_p = 0$), the Hessian form is defined as

$$\begin{aligned} \mathrm{Hess_p}\, f : T_pM \times T_pM &\to \mathbb{R} \\ (v, w) &\mapsto X(Y(f)) \end{aligned}$$

where X and Y are vector fields defined locally near p with $X(p) = v$ and $Y(p) = w$.

a) Prove that this form is indeed a well-defined and symmetric bilinear form.

Siehe nächstes Blatt!

b) The function f is called a *Morse function* if at all critical points the Hessian is non-degenerate, i.e. has all its eigenvalues (in any and hence every coordinate representation) non-zero. Prove that $f : M \to \mathbb{R}$ is Morse if and only if df is a section of T^*M which is transverse to the zero section.

c) Prove that the index of a nondegenerate zero p of df, viewed as a section of T^*M, is the same as the index of the Hessian of f at p, i.e. the number of negative eigenvalues of $\mathrm{Hess}_\mathrm{p} f$.

9. Which closed oriented surfaces admit 1-dimensional foliations?

10. The forms $\lambda_1 = x\,dy + dz$ and $\lambda_2 = dz + \frac{1}{2}(x\,dy - y\,dx)$ on $\mathbb{R}^3$ are examples of contact forms on $\mathbb{R}^3$, in fact they satisfy $\lambda_i \wedge d\lambda_i = dx \wedge dy \wedge dz$, which is nowhere zero.

a) Try to draw the corresponding kernel distributions $E_i = \ker \lambda_i$.

b) Prove that there is a diffeomorphism φ of $\mathbb{R}^3$ such that $\varphi^* \lambda_2 = \lambda_1$.