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Differential Topology

Problem Set 1

Here are some problems related to the material of the course up to this point. Some of them were already mentioned in class. Not all of them are equally difficult, or equally important. If you want to get feedback on your solution to a particular exercise, you may hand it in after any lecture, and I will try to comment within a week.

1. Put a smooth structure on the boundary B of the n-dimensional cube, i.e. the subset

$$B := \{(x_1, \ldots, x_n) \subset \mathbb{R}^n \mid \max_i |x_i| = 1\} \subset \mathbb{R}^n.$$

Is B with this structure a submanifold of $\mathbb{R}^n$? Does the smooth manifold you construct embed smoothly into $\mathbb{R}^n$?

2. Prove that the subset

$$Q := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} z_j^2 = 1\} \subset \mathbb{C}^n$$

is diffeomorphic to the tangent bundle of S^{n-1}.

3. Let M be a differentiable manifold (of some class C^r, $r \geq 1$), and let $\tau : M \to M$ be a fixed point free involution, i.e. $\tau(p) \neq p$ for all $p \in M$ and $\tau \circ \tau = \mathrm{id}_M$.

a) Prove that the quotient space M/τ which is obtained by identifying every point with its image under τ is a topological manifold, and it admits a unique C^r-structure for which the projection map $\pi : M \to M/\tau$ is a local diffeomorphism.

b) Give examples of this phenomenon.

4. Give a proof of the Whitney embedding theorem for noncompact manifolds: Every smooth manifold of dimension n has an embedding as a submanifold into $\mathbb{R}^{2n+1}$ such that the image is a closed subset.

5. Prove directly that every product of spheres of total dimension n can be embedded into $\mathbb{R}^{n+1}$.

6. Prove that if M is a manifold of dimension $d < d'$, then any smooth map $f : M \to S^{d'}$ is homotopic to a constant map. In contrast, construct a map of degree 1 in case $d = d'$ and M is closed (here the degree is taken $\mod 2$ if M is not orientable).

Bitte wenden!

7. Prove that a manifold M is orientable if and only if the restriction of the tangent bundle to every closed curve in M is an orientable vector bundle.

8. Prove that every map $S^d \to S^d$ with degree different from $(-1)^{d+1}$ has a fixed point.

9. Prove that any map $S^d \to S^d$ of odd degree maps some pair of antipodal points onto a pair of antipodal points.

10. Let M be a connected manifold of dimension $d \geq 2$. Prove that given two k-tupels $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ of distinct points in M, there exists a diffeomorphism $\varphi : M \to M$ satisfying $\varphi(x_j) = y_j$ for all $j = 1, \dots k$.

11. Prove that every Lie group has trivial tangent bundle.

12. Prove that for any vector bundle $p : E \to B$, the direct sum $E \oplus E \to B$ is orientable. Deduce as a consequence that the manifold TM is orientable for any smooth manifold M.

13. Let $p : E \to B$ be a vector bundle over a connected base space B and let $F : E \to E$ be a bundle morphism covering the identity on B and satisfying $F \circ F = F$. Prove that F has constant rank, and deduce that $\ker F$ and $\operatorname{Im} F$ are subbundles of E.

14. Construct a vector field on S^2 with exactly one zero.

15. Let M be a differentiable manifold, and let $\Delta \subset M \times M$ be the diagonal submanifold

$$\Delta := \{(p, p) \mid p \in M\} \subset M \times M.$$

Prove that the tangent bundle of Δ and the normal bundle of Δ in $M \times M$ are isomorphic.

16. Let $f : M \to M'$ be a smooth map between manifolds transverse to the submanifold $Z' \subset M'$. We know that $Z := f^{-1}(Z') \subset M$ is a smooth submanifold of the same codimension as Z'. Prove that the normal bundle of Z in M is isomorphic to the pullback via f of the normal bundle of Z' in M'.

17. Let $f_1 : M_1 \to M'$ and $f_2 : M_2 \to M'$ be two smooth maps which are *transverse* in the sense that for every pair of points $(p_1, p_2) \in M_1 \times M_2)$ with $f_1(p_1) = f_2(p_2) =: q'$ one has

$$Df_1(T_{p_1} M_1) + Df_2(T_{p_2} M_2) = T_{q'} M'.$$

Prove that under these conditions the *fiber product of* M_1 *and* M_2 *over* M',

$$M_1 \times_{M'} M_2 := \{(p_1, p_2) \mid f_1(p_1) = f_2(p_2)\} \subset M_1 \times M_2,$$

is a smooth submanifold. What is its dimension? Can you find interesting examples of this construction?

18. Prove that if M is oriented and S_1 and S_2 are oriented closed submanifolds of complementary dimension, then the intersection number satisfies

$$S_2 \bullet S_1 = (-1)^{\dim S_1 \cdot \dim S_2} S_1 \bullet S_2.$$

Use the same argument that the Euler characteristic of an odd dimensional manifold vanishes.

19. Construct an immersion of the punctured torus $S^1 \times S^1 \setminus \{*\}$ into $\mathbb{R}^2$. Can such an immersion be injective?

20. In the lecture we defined the Grassmannian manifolds $G_{k,n}$, and the tautological bundles $\Gamma_{k,n} \to G_{k,n}$. Observe that, by definition, $G_{1,n+1} = \mathbb{R}P^n$.

a) Prove directly that $\Gamma_{1,n+1} \to G_{1,n+1} = \mathbb{R}P^{n+1}$ is not a trivial bundle.

b) Prove that $\Gamma_{1,n}$ is isomorphic to the normal bundle of $\mathbb{R}P^n \subset \mathbb{R}P^{n+1}$. In fact, one can show that the complement of a point $\mathbb{R}P^{n+1} \setminus \{pt\}$ is diffeomorphic to the total space of $\Gamma_{1,n}$.