Introduction to symplectic geometry

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Summary of topics

1. Linear symplectic geometry (treated as background for the main topics)
   (a) the different types of linear subspaces and their normal forms
   (b) linear symplectic maps and their properties
   (c) relations between $\mathrm{Sp}(2n)$, $U(n)$ and $O(2n)$
   (d) Maslov indices for loops of symplectic matrices and of Langrangian subspaces: definitions, properties and computations in examples
   (e) compatible complex structures, contractibility of $\mathcal{J}(\omega_{\mathrm{st}})$
   (f) relation to hermitian metrics
2. Basic on symplectic and contact manifolds
   (a) basic definitions and examples
   (b) symplectic and Hamiltonian diffeomorphisms, inlcuding their basic properties and construction of examples with prescribed behavior (when possible)
   (c) Moser's argument with applications, especially Darboux' theorem
   (d) special submanifolds (isotropic, Lagaŕangian, coisotropic)
   (e) Lagrangian neighborhood theorem
   (f) contact manifolds, symplectization, Reeb flow
   (g) contact vector fields and contact Hamiltonian functions
   (h) Darboux' theorem for contact manifolds, Gray's theorem
   (i) integrable vs. non-integrable complex structures
   (j) Kähler manifolds: definition, examples, Kähler forms as $(1,1)$-forms, Kähler potentials
3. J-holomorphic curves
   (a) definition, equivalent forms of defining equations, energy, examples
   (b) spaces of J-holomorphic curves, statement of regularity
   (c) compactness of the moduli space: basic phenomena (what happens with uniform gradient bounds? what is bubbling?)
   (d) strategy of the proof of the non-squeezing theorem

Advice for your exam preparation

Of course you will need to know the statements and proofs of the main results. Make sure you also know and understand *many* examples. For instance, in the exam I might ask you to write down a function on $\mathbb{R}^{2n}$ whose Hamiltonian flow will have a prescribed effect, or to compute the Maslov index of some explicit loop of Lagrangian subspaces, or give an example of a J-holomoprhic curve with certain properties, or ...

Other questions I like to ask include: What happens to a given theorem when you leave out one of the assumptions? Do you know counterexamples? What is a simple situation where a given theorem is useful? How is it proven?

As you prepare for the exam, look back at the exercises, as they often give a valuable second perspective on topics covered in the lecture. Also, you might find it useful to look up the treatment of the topics we covered in the textbooks by A. Cannas da Silva or by D. McDuff and D. Salamon (or H. Geiges for the part on contact manifolds).

You may start the exam with a topic of your choice from the following list:

- **symplectic vs. Hamiltonian diffeomorphisms, their properties and examples**
- **Moser's method and its applications**
- **J-holomorphic curves**

The duration of the exam is between 20 and 30 minutes.