

# SYMPLECTIC GEOMETRY

## Problem Set 5

1. Suppose  $S \subseteq \mathbb{R}^{n+1}$  is a smooth strictly convex hypersurface (necessarily diffeomorphic to  $S^n$ ). Let

$$\Delta_k := \{(x_1, \dots, x_k) \in S \times \dots \times S \mid \exists 1 \leq i < k : x_i = x_{i+1} \text{ or } x_k = x_1\} \subseteq S \times \dots \times S.$$

- a) Prove that a critical point of the function

$$f_k : S \times \dots \times S \setminus \Delta_k \rightarrow \mathbb{R} \\ (x_1, \dots, x_k) \mapsto |x_2 - x_1| + \dots + |x_k - x_{k-1}| + |x_1 - x_k|.$$

corresponds to a  $k$ -periodic orbit of the billard map for the domain bounded by  $S$ .

- b) Prove that for each  $k \geq 2$  there exists a  $k$ -periodic billard trajectory.  
c) Can you prove the existence of two geometrically distinct  $k$ -periodic trajectories?

*Hint: If you have trouble getting started, you should try the case  $n = 1$ .*