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ARITHMETIC INTERSECTION THEORY ON COMPACTIFIED SHIMURA VARIETIES

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Abstract. We report on recent results in the arithmetic intersection theory on compactified Shimura Varieties.

1. Introduction

The content of this note is based on joint work with J.H. Bruinier, J.I. Burgos and J. Kramer.

1. Motivation. We are interested in Arakelov theory for compactifications of non-compact Shimura varieties where all the analytical data that is required for arithmetic intersection theory is natural. Such a theory must be a generalization of the theory of H. Gillet and C. Soulé [**Sou92**], since the intrinsic metrics have log-log singularities with respect to the boundary. Observe that the existence of such a theory is already used in the following

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1.1. Meta conjecture: *The natural arithmetic intersection numbers of Shimura varieties are essentially given by logarithmic derivatives of L-functions.*

Precise formulations of this philosophy are given by S. Kudla for Shimura varieties of type $O(2,n)$ [**Kud03**], by K. Köhler for the moduli space of abelian varieties $\mathscr{A}_g$ [**Kö1**] and by V. Maillot and D. Rössler for families of (semi) abelian varieties [**MR02**]. Joint work of our group provides evidence for these conjectures (see [**BK03**], [**BBGK03**], [**BGKKb**], [**BGKKa**]).

2. Review of higher dimensional Arakelov theory

Here we collect some of the basic properties of higher dimensional Arakelov theory. For more details and proofs we refer to [**Sou92**]. We let $\pi : X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic variety. For simplicity we assume that X is a projective, regular scheme flat over $\operatorname{Spec}\mathbb{Z}$ whose generic fiber is smooth. We set $d = \dim_{\mathbb{Z}}(X)$. We write $Z^p(X)$ for the free group of p-codimensional cycles on X. It is a fact that for each $Z \in Z^p(X)$ there exists a Green current g_Z, i.e., a current that satisfies

$$\mathrm{dd}^c\, g_Z + \delta_Z = [\omega_Z],$$

where δ_Z denotes the current of integration along $Z(\mathbb{C})$ and $[\omega_Z]$ denotes the current associated with a **smooth** form ω_Z. Then the free group of arithmetic cycles is

$$\widehat{Z}^p(X) = \{(Z, g_Z) \,|\, Z \in Z^p \text{ and } g_Z \text{ a Green current for } Z\}.$$

It contains the subgroup

$$\widehat{Rat}^p(X) = \{(\operatorname{div}(f), g(f)) \,|\, f \text{ a } K_1\text{-chain and } g(f) \text{ a canonical Green current}\}.$$

Finally the p-th arithmetic Chow group (in the sense of Gillet and Soulé) is defined by

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{Rat}^p(X).$$

Some of its properties are:

2.1. There exists an *arithmetic degree* map

$$\begin{tikzcd} \widehat{\mathrm{CH}}^{d+1}(X) \arrow[d, "\pi_*"'] \arrow[dr, "\widehat{\deg}"] & \\ \widehat{\mathrm{CH}}^{1}(\operatorname{Spec}\mathbb{Z}) \arrow[r, "\sim"] & \mathbb{R}. \end{tikzcd}$$

2.2. Given a morphism of arithmetic varieties then there exists a push-forward (for proper, generically smooth morphisms only!) and a pull-back for the associated arithmetic Chow groups.

2.3. Each hermitian line bundle $\overline{L}$, i.e., a pair $\overline{L} = (L, \|\cdot\|)$ consisting of a line bundle L on X and a **smooth** hermitian metric on the induced bundle $L(\mathbb{C})$, defines a first arithmetic Chern class $\widehat{\mathrm{c}}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(X)$.

2.4. There is an arithmetic intersection product

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}},$$

given by $(Y, g_Y) \otimes (Z, g_Z) \mapsto (Y \cdot Z, g_Y * g_Z)$. The right hand side has to be tensored with $\mathbb{Q}$ since the construction of $Y \cdot Z$ involves K-theory. It is quite technical to show that the star product $g_Y * g_Z = g_Z \delta_Y + g_Y \omega_Z$ is well-defined.

2.5. There is a height pairing

$$Z^p(X) \otimes \widehat{\mathrm{CH}}^{d-p+1}(X) \longrightarrow \mathbb{R},$$

given by $Z \otimes \alpha \mapsto \mathrm{ht}_\alpha(Z)$. In particular if $\alpha = \widehat{\mathrm{c}}_1(\overline{L})$, then $\mathrm{ht}_\alpha(Z) = \mathrm{ht}_{\overline{L}}(Z)$ is also refered to as the Faltings height with respect to $\overline{L}$.

3. Cohomological arithmetic Chow groups

We briefly describe the arithmetic Chow groups presented in [**BGKKb**]. The main idea is to replace the Green equation $\mathrm{dd}^c g_Z + \delta_Z = [\omega_Z]$ by a cohomological relation $\mathrm{cl}(g_Z) = \mathrm{cl}(Z)$ in a suitable cohomology theory.

Recall our assumption that $X(\mathbb{C})$ is compact, therefore

$$g_Z \equiv [g_Z] \mod \operatorname{im}\partial + \operatorname{im}\overline{\partial}$$

for a differential form g_Z with logarithmic singularities along $Z(\mathbb{C})$. In particular g_Z is an element of the Deligne algebra $\mathscr{D}^*_{\log}(X \setminus Z, *)$ of smooth differential forms on $X \setminus Z$ with logarithmic singularities along the boundary. Now the first key observation due to J. Burgos [**Bur97**] is that with respect to the natural isomorphisms

$$H^{2p}\big(\mathscr{D}^{2p}_{\log}(X,p), \mathscr{D}^{2p-1}_{\log}(X \setminus Z, p)\big) \cong H^{2p}_{Z,\mathscr{D}}(X,p) \cong H^{2p}_Z(X,\mathbb{R})$$

we get the identifications

$$\mathrm{cl}\left((\omega_Z, g_Z)\right) \widehat{=} \mathrm{cl}(Z) \widehat{=} \delta_Z,$$

here the cohomology group in the middle is the Deligne cohomology group with support in the real cycle associated with Z. The second key observation also due to J. Burgos is that it is possible to interpret the star product

$$g_Y * g_Z = g_Z \delta_Y + g_Y \omega_Z$$

as a product of truncated relative cohomology groups

$$\widehat{H}^{2p}\big(\mathscr{D}^{2p}_{\log}(X,p),\mathscr{D}^{2p-1}_{\log}(X\setminus Y,p)\big)\otimes\widehat{H}^{2q}\big(\mathscr{D}^{2p}_{\log}(X,q),\mathscr{D}^{2q-1}_{\log}(X\setminus Z,q)\big)$$
$$\longrightarrow\widehat{H}^{2p+2q}\big(\mathscr{D}^{2p+2q}_{\log}(X,p+q),\mathscr{D}^{2p+2q-1}_{\log}(X\setminus(Y\cap Z,p+q)\big)$$

Note that this interpretation allows us to replace many of the analytical identities used in the proof of the well-definedness by homological identities.

Instead of the sheaf complexes $U\mapsto\mathscr{D}^*_{\log}(U,*)$ one could consider other sheaves of complexes. We call a sheaf of complexes that receives classes for cycles and K_1-chains together with some mild compatibility assumptions (see [**BGKKb**], Lemma 3.9.) an arithmetic complex on X. One of the main results in [**BGKKb**] may be stated as follows

Theorem 3.1. *Given an arithmetic complex $\mathscr{C}^*(\cdot,*)$ on X, then there exist arithmetic Chow groups $\widehat{\mathrm{CH}}^*(X,\mathscr{C})$ whose properties are dictated by the functorial and multiplicative properties of $\mathscr{C}^*(\cdot,*)$.*

It is shown in [**Bur97**] that $\widehat{\mathrm{CH}}^*(X,\mathscr{D}_{\log})\cong\widehat{\mathrm{CH}}^*(X)$.

4. Arithmetic Chow rings with pre-log-log forms

We let $\pi:X\to\operatorname{Spec}\mathbb{Z}$ be an arithmetic variety as before. In addition we fix a divisor $D\subset X(\mathbb{C})$ with normal crossings. Let Z be a cycle on X and $\mathscr{D}^*_{\mathrm{pre}}(X\setminus Z)$ be the Deligne algebra of pre-log-log forms with respect to Z. Here a pre-log-log form η with respect to Z is a smooth differential form on $\big(X\setminus(Z\cup D)\big)(\mathbb{C})$ such that η, $\partial\eta$, $\overline{\partial}\eta$ and $\partial\overline{\partial}\eta$ have logarithmic singularities along Z and log-log singularities along D. We have

Theorem 4.1. *The cohomological arithmetic Chow ring*

$$\widehat{\mathrm{CH}}^*(X,\mathscr{D}_{pre})_{\mathbb{Q}}=\bigoplus_p\widehat{\mathrm{CH}}^p(X,\mathscr{D}_{pre})_{\mathbb{Q}}$$

is a non trivial extension of $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$.

The arithmetic Chow group $\widehat{\mathrm{CH}}^*(X,\mathscr{D}_{\mathrm{pre}})$ is an extension since $\mathscr{D}_{\log}$ is a sub-complex of $\mathscr{D}_{\mathrm{pre}}$. Since pre-log-log forms with respect to $\emptyset$ are integrable there is an arithmetic degree map

$$\widehat{\deg}:\widehat{\mathrm{CH}}^{d+1}(X,\mathscr{D}_{\mathrm{pre}})\longrightarrow\mathbb{R}.$$

Any line bundle L on X equipped with a logarithmically singular hermitian metric on $L(\mathbb{C})$ with respect to D determines a class in $\widehat{\mathrm{c}}_1(\overline{L})\in\widehat{\mathrm{CH}}^1(X,\mathscr{D}_{\mathrm{pre}})$

that is called the first arithmetic Chern class. We have an arithmetic intersection pairing. There is also a height pairing. More precisely if

$$Z^p_U(X) = \{Z \in Z^p(X) \mid Z(\mathbb{C}) \text{ intersects } D \text{ properly}\},$$

then there is a well defined pairing

$$Z^p_U(X) \otimes \widehat{\mathrm{CH}}^{d-p+1}(X, \mathscr{D}_{\mathrm{pre}}) \longrightarrow \mathbb{R},$$

given by $Z \otimes \alpha \mapsto \mathrm{ht}_\alpha(Z)$. If $\alpha = \widehat{\mathrm{c}}_1(\overline{L})^{d-p+1}$, then this height is also refered to as Faltings height; because the particular case $X = \mathscr{A}_g$, $\overline{L} = \overline{\mathscr{M}}$ the line bundle of modular forms with its Petersson metric and $p = d$ yields exactly the modular height of abelian varieties as it was used by Faltings in his proof of the Mordell conjecture.

Finally we remark that if our arithmetic variety X is a compactified Shimura variety, i.e., X is an arithmetic variety such that $X(\mathbb{C}) = \overline{\Gamma \setminus G/K}$ where Γ is a discrete subgroup in the automorphism group of a bounded symmetric domain G/K, $D = X(\mathbb{C}) \setminus \big(\Gamma \setminus G/K\big)$ and $\overline{L}$ is an automorphic line bundle equipped with a $G(\mathbb{R})$-invariant metric, then our results of **[BGKKb]**,**[BGKKa]** apply. In particular they imply that $\widehat{\deg}\big(\widehat{\mathrm{c}}_1(\overline{L})^{d+1}\big)$ is a well-defined number. Numbers of this type are called *arithmetic intersection numbers* on X.

It is known that the geometric intersection numbers for such L are given by special values of L-functions and zeta functions. The above conjecture is that the arithmetic intersection numbers for $\overline{L}$ are essentially given by same special values but now of the logarithmic derivative of the same L-functions and zeta functions.

5. Explicit calculations

The above theory has been proved to be applicable to the following cases.

5.1. Modular curves. For simplicity we consider the elliptic modular curve $\pi : X(1) \to \operatorname{Spec}\mathbb{Z}$ and its line bundle of modular forms $\overline{\mathscr{M}}_k$ of weight k equipped with the Petersson metric. It is well known that $X(1)(\mathbb{C}) = \overline{\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2}$ and that the global sections of $\mathscr{M}_k(\mathbb{C})$ are holomorphic modular forms of weight k. In 1998 the author and independently also J.-B. Bost proved the following

Theorem 5.1. **[Küh01]** *Let $\zeta_{\mathbb{Q}}(s)$ be the Riemann zeta function, then the arithmetic self intersection number of the line bundle of modular forms with its*

Petersson metric is given by

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{M}}_k)^2\big) = k^2\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

5.2. Products of modular curves. We consider the arithmetic threefold $H = X(1) \times X(1)$ and on it the line bundle of bi-modular forms $\overline{L}_k = p_1^*\overline{\mathscr{M}}_k \otimes p_2^*\overline{\mathscr{M}}_k$ of weight k. Then functoriality and the previous result imply

Theorem 5.2. [**BGKKb**] *The arithmetic self intersection number of the line bundle $\overline{L}_k$ of bi-modular forms on H is given by*

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{M}}_k)^3\big) = \frac{k^3}{2}\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

5.3. Hilbert modular surfaces. Let K be a real quadratic number field. We write $\mathscr{O}_K$ for its ring of integers. The complex surface $\mathrm{SL}_2(\mathscr{O}_K) \setminus \mathbb{H}^2$ is called the Hilbert modular surface associated to K. Let $\overline{\mathrm{SL}_2(\mathscr{O}_K) \setminus \mathbb{H}^2}$ be a compactified desingularisation of it. Assume that the discriminant D_K of K is a prime congruent to 1 modulo 4. To ease notation we assume that there exists an arithmetic variety $\pi : H \to \operatorname{Spec}\mathbb{Z}$ such that $H(\mathbb{C}) = \overline{\mathrm{SL}_2(\mathscr{O}_K) \setminus \mathbb{H}^2}$. It is not known whether such an arithmetic threefold exist. However, if we consider certain congruence subgroups of $\mathrm{SL}_2(\mathscr{O}_K)$, then there exists such arithmetic threefolds over certain subrings of cyclotomic fields.

Theorem 5.3. [**BBGK03**] *Under the above simplifying assumption, the arithmetic self intersection number of the line bundle $\overline{\mathscr{M}}_k$ of Hilbert modular forms on H is given by*

$$\widehat{\deg}\big(\widehat{c}_1(\overline{\mathscr{M}}_k)^3\big) = k^3\zeta_K(-1)\left(\frac{\zeta'_K(-1)}{\zeta_K(-1)} + \frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{3}{2} + \frac{1}{2}\log(D_K)\right),$$

where $\zeta_K(s)$ is the Dedekind zeta function and D_K is the discriminant of K.

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