Riemannian geometry

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Exercise 1

Let M be a Riemannian manifold. For a vector field X on M, we define the Lie-Derivative of the connection as follows:

$$(L_X\nabla)(Y,Z) = L_X(\nabla_Y Z) - \nabla_{L_X Y}Z - \nabla_Y L_X Z = [X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z]$$

a) Show that $L_X\nabla$ is $C^\infty(M)$-linear in both arguments.

b) We say that X is an affine vector field if $L_X\nabla = 0$. Show that for such a field, we have

$$\nabla^2_{Y,Z}X = -R(X,Y)Z$$

Hint: Show that $R(W,Y)Z + \nabla^2_{Y,Z}W = (L_W\nabla)(Y,Z)$ *for all* $X,Y,Z \in \mathfrak{X}(M)$.

c) Show that any Killing vector field is affine. Give an example of an affine vector field on $\mathbb{R}^n$ that is not Killing.

Exercise 2

Let $f : M \to \mathbb{R}$ be a continuous function on a complete connected manifold M and suppose that $H(f) \geq 0$ everywhere in the sense of Definition 4.8.

a) Show that f cannot have a global maximum on M unless it is constant (without using the strong maximum principle). *Hint: Show that* $f \circ \gamma$ *is convex for each geodesic* γ.

b) Show that if f is linear, we have for each geodesic γ that $f \circ \gamma(t) = at + b$ for some $a, b \in \mathbb{R}$. Prove that for each $p \in M$ there exists a $w_p \in T_pM$ such that $f \circ \exp_p(v) = f(p) + \langle w_p, v\rangle$ for all $v \in T_pM$. Conclude that f is smooth and show that $\mathrm{grad} f_p = w_p$.

Exercise 3

Show the following fact to complete the proof of the Cheeger-Gromoll splitting theorem: If (M,g) is a complete and connected Riemannian manifold and $f : M \to \mathbb{R}$ is linear, then (M,g) splits isometrically as $(N \times \mathbb{R}, h + dt^2)$. *Hint: Consider the level sets of the function* f*. Use the fact that* $\mathrm{grad} f$ *is a Killing field (why?) and therefore generates a one-parameter family of isometries.*

Exercise 4

Let M be a complete Riemannian manifold, $p \in M$ and $f : T_1M \to \mathbb{R}_+ \cup \{\infty\}$ be the function introduced in Chapter 10.

a) Show that

$$\begin{aligned} M \setminus C_m(p) &= \{\exp_p(tv) \mid t < f(p,v), v \in T_pM, \|v\| = 1\} \\ C_m(p) &= \{\exp_p(tv) \mid t = f(p,v) < \infty, v \in T_pM, \|v\| = 1\} \\ M &= \{\exp_p(tv) \mid t \leq f(p,v), v \in T_pM, \|v\| = 1\} \end{aligned}$$

b) Show that $M \setminus C_m(p)$ is contractible.

c) Show that $C_m(p)$ is closed.