Riemannian geometry

Winter term 2015/16

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Exercise 1

Let $\gamma : [0,1] \to M$ be a geodesic and a Riemannian manifold M, J be a Jacobi field along γ vanishing at the origin and X the parallel translation of $J'(0)$ along γ. For $p \in M$ and $v \in T_pM$ with $\|v\| = 1$, let $J(v,t)$ be the function introduced before Example 5.5. Show that

$$J(t) = tX - \frac{t^3}{6} R(X, \gamma')\gamma' + o(t^3),$$
$$J(v,t) = 1 - \frac{t^2}{6}\mathrm{Ric}(v,v) + o(t^2).$$

Hint: Use (2.15.1) *in the script of Oliver Goertsches.*

Exercise 2

a) Let ϕ be a symmetric bilinear form on $\mathbb{R}^n$. Use a diagonal basis to show

$$\int_{S^{n-1}} \phi(v,v)dv = \frac{1}{n}\mathrm{vol}(S^{n-1})\mathrm{tr}(\phi).$$

b) Let p be a point in a complete n-dimensional Riemannian manifold. Let $B_r(p)$ be the ball of radius p around p and $\tilde{B}_r$ be the ball of radius r in the flat euclidean space $\mathbb{R}^n$. Use Exercise 1 to show that

$$\mathrm{vol}(B_r(p)) = \mathrm{vol}(\tilde{B}_r)(1 - \frac{\mathrm{scal}(p)}{6(n+2)}r^2 + o(r^2)).$$

Exercise 3

a) Show that any odd-dimensional Riemannian manifold of positive sectional curvature is orientable.
Hint: Adapt the proof of Theorem 6.5 and use the fact that any matrix $A \subset O(2m) \setminus SO(2m)$ admits 1 *as an eigenvalue*

b) For which $n \in \mathbb{N}$ is $\mathbb{R}P^n$ orientable resp. not orientable?

Exercise 4

A group G is called solvable if there exists a finite sequence of subgroups

$$\{e\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_k = G$$

such that G_{l-1} is normal in G_l and G_l/G_{l-1} is abelian for $1 \leq l \leq k$.
Generalize Theorem 7.12 to the following statement: If M is a compact manifold of negative sectional curvature, then any solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.
Hint: Show that if G_{l-1} is isomorphic to $\mathbb{Z}$ and $\tilde{\gamma}$ is a geodesic in $\tilde{M}$ invariant under all elements of G_l, then $\tilde{\gamma}$ is also invariant under all elements of G_l.