Riemannian geometry

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Klaus Kröncke

Exercise sheet 4 — Dec 4, 2015

Exercise 1

Let M, N be Riemannian manifolds and $M \times N$ the product manifold. For any vector field X on $M \times N$, we write $X = (X_1, X_2)$ according to the canonical decomposition $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$. We equip $M \times N$ with the product metric, given by $\langle X, Y \rangle_{M\times N} = \langle X_1, Y_1 \rangle_M + \langle X_2, Y_2 \rangle_N$. Show that

$$\begin{aligned} R^{M\times N}(X,Y)Z &= R^M(X_1,Y_1)Z_1 + R^N(X_2,Y_2)Z_2. \\ \mathrm{Ric}^{M\times N}(X,Y) &= \mathrm{Ric}^M(X_1,Y_1) + \mathrm{Ric}^N(X_2,Y_2), \\ \mathrm{scal}^{M\times N} &= \mathrm{scal}^M + \mathrm{scal}^N. \end{aligned}$$

Prove that if the sectional curvatures on M and N both satisfy $K \geq 0$ (resp. $K \leq 0$) then the same holds for the sectional curvature on $M \times N$. Show that a product manifold always has planes of zero sectional curvature.

Exercise 2

Let $M \times N$ be as above.

a) Show that a curve $\gamma = (\gamma_1, \gamma_2) : [0, a] \to M \times N$ is a geodesic if and only if γ_1 and γ_2 are geodesics in M and N, respectively.

b) Let $\gamma = (\gamma_1, \gamma_2)$ be a geodesic in $M \times N$. Show that γ is minimizing if and only if γ_1 and γ_2 are minimizing.

c) Let $p = (p_1, p_2) \in M \times N$ and $C_m(p)$ be the cut locus of p. Show that $C_m(p) = C_m(p_1) \times N \cup M \times C_m(p_2)$

d) Show that for $p = (p_1, p_2), q = (q_1, q_2) \in M \times N$, the distance is given by

$$d^{M\times N}(p,q) = \sqrt{d^M(p_1,q_1) + d^N(p_2,q_2)}.$$

Exercise 3

In this exercise, $\mathbb{R}$ is always equipped with the standard metric.

a) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. Show that $H(f)_0 \geq C$ in the sense of Definition 4.8 for any $C \in \mathbb{R}$.

b) Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

Show that $H(f)_0 \geq 0$ in the sense of Definition 4.8 and that $H(f)_0 \geq C$ does not hold if $C > 0$.

Exercise 4

Let M^n be a Riemannian manifold such that the sectional curvature K violates the condition $K \geq \kappa$. Show then that there exists a hinge $\{\gamma_1, \gamma_2\}$ and a comparison hinge $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ in M_κ^2 (the simply connected space form of dimension 2 and curvature κ) such that

$$d(\gamma_1(L_1), \gamma_2(L_2)) > d(\tilde{\gamma}_1(L_1), \tilde{\gamma}_2(L_2)).$$

Hint: Let $p \in M$ and $\sigma \subset T_pM$ be a plane such that $K(\sigma) < \kappa$. Let $\{\gamma_1, \gamma_2\}$ be two normalized geodesics starting in p such that $\operatorname{span}\{\gamma_1'(0), \gamma_2'(0)\} = \sigma$*. Let α the minimizing geodesic joining the endpoints of the hinge and use Proposition 2.9 to construct a comparison hinge $\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ in M_κ^n and a curve $\tilde{\alpha}$ shorter than α connecting the endpoints of this hinge. Why can we descend to M_κ^2?*