Riemannian geometry

Winter term 2015/16

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Exercise sheet 3 — Nov 20, 2015

Exercise 1

Consider the two ordinary differential equations

$$f''(t) + K(t)f(t) = 0, \qquad f(0) = 0, \qquad t \in [0, a]$$
$$\tilde{f}''(t) + \tilde{K}(t)\tilde{f}(t) = 0, \qquad \tilde{f}(0) = 0, \qquad t \in [0, a].$$

Suppose that $\tilde{K}(t) \geq K(t)$ and $f'(0) = \tilde{f}'(0) = 1$.

a) Show that

$$0 = [\tilde{f}f' - f\tilde{f}']_0^t + \int_0^t (K - \tilde{K}) f\tilde{f} dt$$

for any $t \in (0, a]$ and conclude that $f > 0$ on $(0, t_0]$ if $\tilde{f} > 0$ on $(0, t_0]$, $t_0 \in (0, a]$.

b) Suppose that $\tilde{f} > 0$ on $(0, a]$. Show that $f \geq \tilde{f}$ on $[0, a]$ and that if $f(t_0) = \tilde{f}(t_0)$ for some $t_0 \in (0, a]$, then $K = \tilde{K}$ on $[0, t_0]$. Show that this implies the Rauch comparison theorem in the case that M and $\tilde{M}$ are both 2-dimensional. *Hint: Use part a) to conclude $f'/f \geq \tilde{f}'/\tilde{f}$. Then show that the function $f/\tilde{f}$ is nondecreasing*

Exercise 2

Let M be a complete manifold whose sectional curvature K satisfies $L \leq K \leq H$, $L, H \in \mathbb{R}$.

a) Let $\gamma : [0, a] \to M$ be a normalized geodesic and suppose that $a \leq \frac{\pi}{\sqrt{H}}$ if $H > 0$. Let J be a Jacobi field along γ such that $\langle J, \gamma' \rangle = 0$. Use Rauch's theorem to show that $\mathrm{sn}_H(t) \, \|J'(0)\| \leq \|J(t)\| \leq \mathrm{sn}_L(t) \, \|J'(0)\|$.

b) Let γ and a be as in part a) and J be a Jacobi field along γ such that $\langle J, \gamma' \rangle$. Show that

$$\frac{\mathrm{cn}_H(t)}{\mathrm{sn}_H(t)} \, \|J(t)\|^2 \leq \langle J'(t), J(t) \rangle \leq \frac{\mathrm{cn}_L(t)}{\mathrm{sn}_L(t)} \, \|J(t)\|^2$$

for all $t \in (0, \infty)$ if $H \leq 0$ or $t \in (0, \frac{\pi}{\sqrt{H}})$ if $H > 0$. *Hint: Consider the functions v'/v and $\tilde{v}'/\tilde{v}$ in the proof of Rauch's theorem and use formula (2.6) of the lecture.*

Exercise 3

The covariant derivative of a one form ω defined such that the product rule $X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$ holds for all vector fields X, Y. This justifies the notion $\nabla_X df$ in Definition 4.4. Let $f : M \to \mathbb{R}$ be a smooth function on the manifold M.

a) Show that the various definitions of $H(f)$ coincide and that it is symmetric.

b) Compute coordinate expressions of $\mathrm{grad} f$ and $H(f)$.

c) Show that for any smooth curve $\alpha : [0, a] \to M$, we have the formula $(f \circ \alpha)''(t) = H(f)(\alpha'(t), \alpha'(t)) + df(\frac{\nabla}{dt}\alpha'(t))$.

Exercise 4

Let M_κ be a complete Riemannian manifold of constant curvature κ, $p \in M$ and U be a normal neighbourhood of p in M_κ. Let $\varphi_p(q) = \frac{1}{2}d(p,q)^2$ and $r(q) = d(p,q)$ be functions on M_κ. Prove that the formulas in Example 4.6 are correct, i.e.

a) compute the gradient of r on $U \setminus \{p\}$,

b) compute the Hessian of φ_p on U and,

c) compute the Hessian of r on $U \setminus \{p\}$.