Riemannian geometry

Winter term 2015/16

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Exercise sheet 2 — Nov 6, 2015

Exercise 1

Let $\gamma : [0,a] \to M$ be a geodesic in the Riemannian manifold M. Show that

a) The vector field $J(t) = (at+b)\gamma'(t)$ is a Jacobi field along γ.

b) A vector field of the form $J(t) = f(t)\gamma'(t)$ is a Jacobi field if and only if it is for the form as in a).

c) A Jacobi field J is orthogonal to γ' for any $t \in [0,a]$ if and only if $J(0)$ and $J'(0)$ are orthogonal to $\gamma'(0)$.

Exercise 2

Let (M,g) be a complete Riemannian manifold of constant curvature $\kappa \in \mathbb{R}$. Let $p \in M$ and V be a normal neighbourhood of p in M. Show that in exponential coordinates $(\exp_p)^{-1} : V \to T_pM = \mathbb{R}^n$, g has the forn

$$\exp_p^* g = dr^2 + (\mathrm{sn}_\kappa(r))^2 g_{S^{n+1}},$$

where r denotes the radial coordinate and $g_{S^{n+1}}$ is the round metric of the $(n-1)$-sphere of radius 1. Here,

$$\mathrm{sn}_\kappa(r) = \begin{cases} \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}, & \text{if } \kappa > 0 \\ r, & \text{if } \kappa = 0 \\ \frac{\sinh(\sqrt{\kappa}r)}{\sqrt{\kappa}}, & \text{if } \kappa < 0. \end{cases}$$

Hint: Use Beispiel 2.14.5 (4) and Satz 2.14.6 in the script of Oliver Goertsches.

Exercise 3

Let $\mathrm{Iso}(M)$ be the Isometry group of the Riemannian manifold M. Show that

$$\begin{aligned} \mathrm{Iso}(S^n) &= \mathrm{O}(n+1) \\ \mathrm{Iso}(\mathbb{R}^n) &= \{x \mapsto Ax + b \mid A \in \mathrm{O}(n), b \in \mathbb{R}^n\} \\ \mathrm{Iso}(H^n) &= \ll X, Y \gg, \end{aligned}$$

where X is the subgroup of maps in $\mathrm{Iso}(\mathbb{R}^n)$ that fixes the last coordinate, Y is the group of isometries of the form as in exercise 3 (a) of sheet 1 and $\ll X, Y \gg$ is the group generated by X and Y.

Exercise 4

Consider the 3-dimensional sphere as the set

$$S^3 = \left\{z_1, z_2 \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\right\}$$

and let $h : S^3 \to S^3$ be given by

$$h(z_1, z_2) = (e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i r}{q}} z_2)$$

where r, q are relative prime numbers

a) Show that $G = \left\{\mathrm{id}, h, \ldots, h^{q-1}\right\}$ is a group of isometries that act totally discontinuous on S^3. The manifold S^3/G is called a lens space.

b) Consider the manifold S^3/G (called the "lens space") with the metric induced by the projection $\pi : S^3 \to S^3/G$ and show that all geodesics of S^3/G are closed but can have different lenghts