Riemannian geometry

Winter term 2015/16

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Exercise 1

Let (M, g) be a Riemannian manifold. Show that the Christoffel symbols of the Levi-Civita connection with respect to a chart are given by the formula

$$\Gamma_{ij}^k = \sum_l \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

We define the local coefficients of the Riemannian curvature tensor with respect to a chart by $R(\partial_i, \partial_j)\partial_k = \sum_l R_{ijk}^l \partial_l$. Prove that

$$R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \sum_m (\Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l).$$

Exercise 2

Show that $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ with the metric $g_{ij} = \frac{1}{(x_n)^2}\delta_{ij}$ is of constant curvature -1 (δ_{ij} denotes the euclidean metric of $\mathbb{R}^n$). Give an example of a metric of constant curvature $-\kappa < 0$.

Exercise 3

a) Consider the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, given (via the identification $\mathbb{R}^2 \cong \mathbb{C}$) by $z \mapsto \frac{az+b}{cz+d}$ (where $ad - bc = 1$). Show that $\mathrm{id}_{\mathbb{R}^{(n-2)}} \times \varphi$ is an isometry of (H^n, g).

b) Show that for $R > 0$ and $C \in \mathbb{R}$, the lines

$$\{(0, \ldots, 0, C, t) \mid t > 0\}$$
$$\{(0, \ldots, 0, R\cos(t) + C, R\sin(t)) \mid t \in (0, \pi)\}$$

are images of geodesics of (H^n, g). Give explicit parametrizations of these geodesics. (Hint: Use part a))

Exercise 4

Show that (H^n, g) is complete. (Hint: Observe that any euclidean motion $x \mapsto Ax + b$, $A \in \mathrm{O}(n), b \in \mathbb{R}^n$ which fixes the x_n-coordinate is an isometry of (H^n, g) and use the previous exercise.)