An integrated model for performance and dependability analysis for an $M/M/1/\infty$ system

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Outline

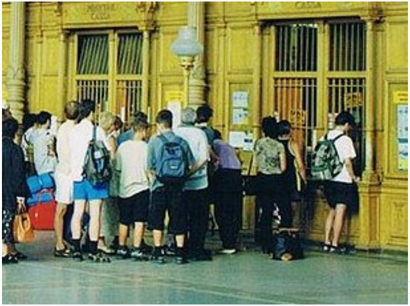
1. Model
   - Focus of research
   - Case study $M/M/1/\infty$ in random environment
   - Interaction between queuing system and environment.
2. Unreliable $M/M/1/\infty$ queuing system.
   - Problem
   - Cost function
   - Solution
3. Numerical examples
4. General results

Queuing system and environment

Our focus of research are stochastic Processes $(X(t), Y(t) : t \in \mathbb{R}_0^+)$,

- $X(t) \in \mathbb{N}_0$ state of the queue (number of customers) at time t.
- $Y(t) \in K$ is a random environment with countable set K at time t, e.g. availability status of the server.
- "Interaction rules" between $X(t)$ and $Y(t)$.

Queuing system in random enviroment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Component: $M/M/1/\infty$ queuing system

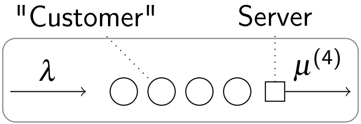


Figure: $M/M/1/\infty$ queuing system model

Mathematical model: a strong Markov process $(X(t) : t \in \mathbb{R}_0^+)$, $X(t) \in \mathbb{N}_0$
System parameters:

- Exponential service rates $\mu^{(n)}$.
- Waiting area of infinite size.
- Poisson input with rate a λ.
- Single server FCFS service policy.
- $X(t)$ describes number of customers in the system at time t.

Component: environment process.

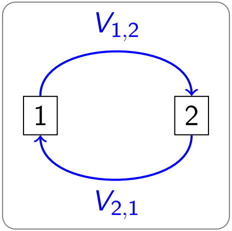


Figure: Example. Environment process.

Mathematical model: A strong Markov process $(Y(t) : t \in \mathbb{R}_0^+)$, $Y(t) \in K$
System parameters:

- Countable state space K.
- Exponential dwell times.
- The system is defined by means of transition rates $V \in \mathbb{R}^{K \times K}$.

Interaction: the environment controls the server

Control with blocking set $K_B \subset K$

- If $Y \in K_B$ the server is blocked ($\hat{=}$down and a new arrivals are *lost*).

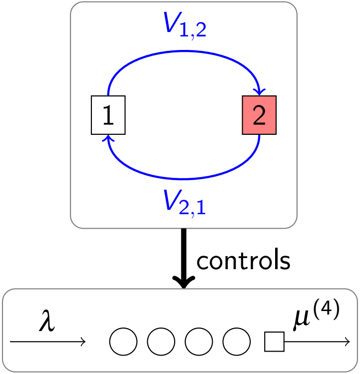


Figure: Example. Environment controls the server, $K_B = \{2\}$.

Interaction: the server controls the environment

Control with stochastic matrix R.

- When a service is finished the environment state k may change instantaneously to m with probability R_{km}.
- R is a stochastic matrix.

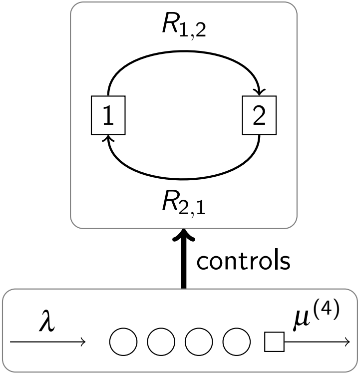


Figure: Example. The server controls the environment. $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Interaktion: server and environment two-way control

Two-way control

It is possible to mix both control types:

- The environment controls queuing system via blocking states K_B.
- Queuing system controls the environment with stochastic matrix R.

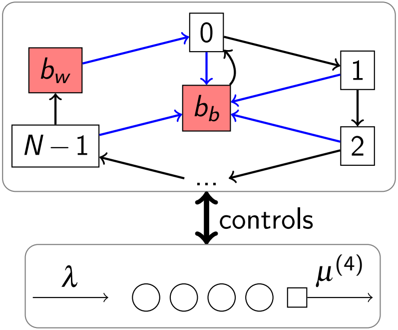


Figure: Example. Two-way control. Blue arrows represent positive rates V, black arrows represent positive entries of the stochastic matrix R.

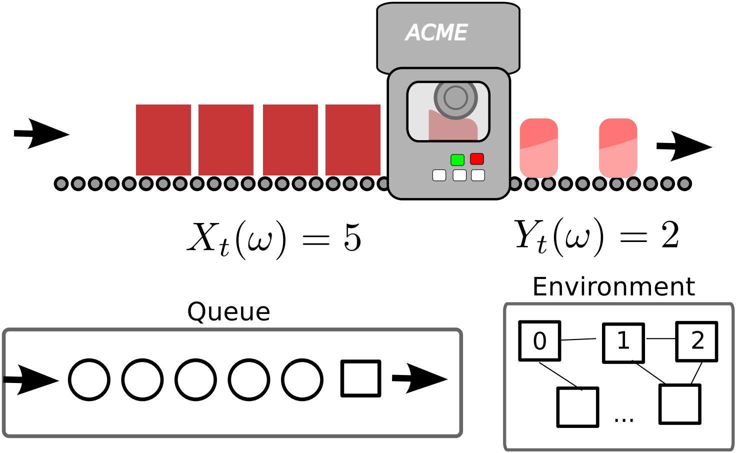
Problem description

We will analyze a queuing system, where the server wears down during service. As a consequence the failure probability increases. After the system breaks down it is repaired and thereafter resumes work as good as new. To prevent break downs, the system will be maintained after a fixed maximal number of services since the most recent repair or maintenance. During repair or maintenance the system is blocked, i.e., no service is provided and no new job may join the system. These rejected jobs are *lost* to the system.

Subject to optimization

- N - the optimal number of services, after which system needs to be maintained.

Modelling



The queuing system

We consider a production system which is modeled as an $M/M/1/\infty$ queuing system.

- λ Poisson input rate.
- The server rates $\vec{\mu} := (\mu^{(n)} : n \in \mathbb{N})$.
- *FCFS* service regime.

The environment

- Environment states $K = K_W + K_B$ with
  - $K_W = \{0, 1, ...N-1\}$ describes the "service counter": number of completed services. N is maximal number of services before maintenance.
  - $K_B = \{b_m, b_r\}$, additional environment states for maintenance and repair.
- Stochastic matrix $R \in [0,1]^{K \times K}$ models the "service counter" behavior.
  - $R_{k,k+1} = 1$, for $0 \leq k \leq N-2$, counter increment.
  - $R_{N-1,b_m} = 1$, maintenance after N services.
- Infinitesimal generator $V \in \mathbb{R}^{K \times K}$ with
  - $V_{k,b_r} = \nu_k$, failure rate after k complete services.
  - $V_{b_m,0} = \nu_m$, maintenance rate.
  - $V_{b_r,0} = \nu_r$, repair rate.

Cost function

- c_m maintenance costs per unit of time.
- c_r repair costs per unit of time.
- c_b costs of non-availability per unit of time.
- c_w waiting costs per customer per unit of time.

Considering the cost function per time unit and state

$$f(n,k) = \begin{cases} c_w \cdot n + c_b + c_m, & k = b_m, \\ c_w \cdot n + c_b + c_r, & k = b_r, \\ c_w \cdot n, & k \in K \backslash \{b_m, b_r\}. \end{cases}$$

n - customer number, k-environment state (service counter, maintenance or repair)

Average costs

The asymptotic average costs for an ergodic system can be calculated as

$$\bar{f}(N) = \frac{1}{T}\int_0^T f\left(X_t(\omega), Y_t(\omega)\right) dt \xrightarrow{T\to\infty} \sum_{(n,k)} f(n,k)\pi(n,k), \qquad P.a.s$$

- $f(n,k)$ cost of the system state (n,k) per unit of time.
- $\pi(n,k)$ steady state probability of the system state (n,k).

Steady state solution

$$P(X=n, Y=k) =: \pi(n,k) = \xi(n)\theta(k), \text{ with}$$

$$\xi(n) := \prod_{i=1}^{n} \frac{\lambda}{\mu^i}\xi(0), \text{ and}$$

$$\theta_N(k) = \prod_{i=1}^{k} \left(\frac{\lambda}{\nu_i+\lambda}\right)^i \theta(0) \qquad 0 \le k \le N-1$$

$$\theta_N(b_m) = \frac{\lambda}{\nu_m}\theta(N-1) = \frac{\lambda}{\nu_m}\prod_{i=1}^{N-1}\left(\frac{\lambda}{\nu_i+\lambda}\right)^i \theta(0)$$

$$\theta_N(b_r) = \left(\frac{(\nu_0+\lambda)}{\nu_r} - \frac{\lambda}{\nu_r}\prod_{i=1}^{N-1}\left(\frac{\lambda}{\nu_i+\lambda}\right)^i\right)\theta(0)$$

Average costs

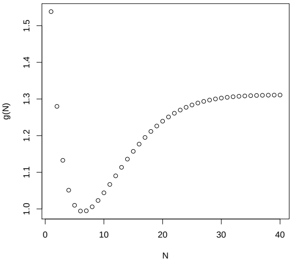
Using product form properties of the system we get

$$\begin{aligned}\bar{f}(N) &= (c_b + c_m)\,\theta_N(b_m) \\ &\quad + (c_b + c_r)\,\theta_N(b_r) \\ &\quad + \underbrace{c_w \sum_{n=1}^{\infty} n\xi(n)}_{\text{independent from } N}\end{aligned}$$

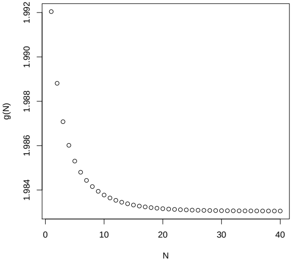
$$\Longrightarrow \arg\min(\bar{f}(N)) = \arg\min(g(N))$$

$$g(N) := (c_b + c_m)\,\theta_N(b_m) + (c_b + c_r)\,\theta_N(b_r)$$

Numerical examples



(a) $v_k = 0.01k$



(b) $v_k \equiv 0.195$

Figure: Cost functions g with $\lambda = 1$, $c_m = 1$, $c_b = 1$, $c_r = 2$, $\nu_m = 0.3$, $\nu_r = 0.1$, $\max(N) = 40$

Numerical example with v_k linear

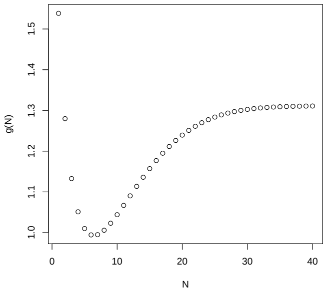


Figure: Cost function g with $0 \leq N \leq 40$, $\lambda = 1$, $c_m = 1$, $c_r = 2$, $c_b = 1$, $\nu_k = 0.01k$, $\nu_m = 0.3$, $\nu_r = 0.1$. Optimal $N = 6$. $g(6) = 0.9943572$, $g(4)/g(40) \approx 0.7585529$

Numerical example with $v_k \equiv const$

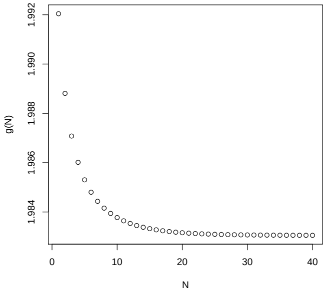


Figure: Cost function g with $0 \leq N \leq 40$, $\lambda = 1$, $c_m = 1$, $c_r = 2$, $c_b = 1$, $\nu_k \equiv \frac{1}{max(N)} \sum_{k=0}^{max(N-1)} 0.01k = 0.195$, $\nu_m = 0.3$, $\nu_r = 0.1$. Optimal $N = 40$. $g(40) = 1.98305$,

Mathematical model

We call a *loss system* [Krenzler and Daduna(2012)] a two dimensional process $(X(t), Y(t) : t \geq 0)$ which describes an ergodic $M/M/1/\infty$ system in random environment with infinitesimal generator q

$$
\begin{aligned}
q((n,k) \to (n+1,k)) &= \lambda, \qquad k \notin K_B \\
q((n,k) \to (n-1,m)) &= \mu^{(n)} R_{km}, \qquad k \notin K_B \\
q((n,k) \to (n,m)) &= V_{k,m} \in \mathbb{R}_0^+, \, k \neq m \\
q((n,k) \to (i,m)) &= 0, \qquad \text{otherwise for } (n,k) \neq (i,m).
\end{aligned}
$$

with environment states K, blocking subset $K_B \subset K$, infinitesimal generator $V \in \mathbb{R}^{K \times K}$ and stochastic matrix $R \in \mathbb{R}^{K \times K}$.

Steady state solution

The loss system has a steady state distribution of product form

$$P(X = n, Y = k) := \pi(n, k) = \xi(n)\theta(k)$$

with

$$\boxed{\xi(n) = \prod_{i=1}^{n} \frac{\lambda}{\mu^i} \xi(0)}$$

and θ, which is a stochastic solution of

$$\boxed{\theta(\lambda(R - I) + V) = 0}$$

Bibliography

Ruslan Krenzler and Hans Daduna.
Loss systems in a random environment - steady state analysis, November 2012.
URL http://preprint.math.uni-hamburg.de/public/papers/prst/prst2012-04.pdf.