Queueing systems in a random environment with applications

Ruslan Krenzler, Hans Daduna

Universität Hamburg
Fachbereich Mathematik

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Queues as a mathematical model

Queueing system

Environment

attendance of employee
has a break / is present

finite buffer
packets in buffer

maintenance status
maintained / ready to use

abstract process
countable state space

Figure: Queueing system examples.
Stochastic model

- countable system states $\mathcal{E} = \mathbb{N}_0 \times K$
  - $\mathbb{N}_0$ queue states (number of customers)
  - $K$ environment state space
- time $t \in [0, \infty]$
- stochastic process $(X(t), Y(t)) \in \mathcal{E}$
  - $X(t)$ number of customers at time $t$
  - $Y(t)$ environment state at time $t$
- exponential sojourn times
- transition rates
- Find: limiting distribution (long term behavior)
  $$\pi(n, k) := \lim_{t \to \infty} P((X(t), Y(t)) = (n, k))$$
- Ansatz: solve $\pi Q = 0$ with generator matrix $Q$ containing the transition rates.
Given: states \((n, k) \in \mathcal{E}\) and transition rates \(Q_{(n,k),(i,m)} \in \mathbb{R}_0^+\)

Find: \(\pi(n, k) := \lim_{t \to \infty} P((X(t), Y(t)) = (n, k))\)

Solve: \(\pi Q = 0, \|\pi\|_1 = 1\)

**Challenge**

- Problem: matrix \(Q\) is large.
  - For a queue with 99 places and 4 environment states we have \(Q \in \mathbb{R}^{400 \times 400}\).
  - For a queue with \(\infty\) capacity we have \(Q \in \mathbb{R}^{\infty \times \infty}\). This system can be easier to solve than one with finite capacity!
Toy problem

Queue at a soft drink vending machine

- Service time is random. Includes: feeding the machine with coins, fetching the can, and so on.
- Service according to FCFS policy.
- Capacity of the machine is limited (maximal three cans).
- As soon as the machine has only 1 can, replenishment is ordered.
- Customer behavior when machine is empty:
  - Customers that were already in the queue, are waiting until replenishment will be finished.
  - New customers go somewhere else \( \triangleq \text{are lost} \).

Find:

Limiting distribution of customers and cans in the vending machine.
Mathematical model

- States \((n, k)\): \(n\) people in queue, \(k\) cans in vending machine. That is \(E = \mathbb{N}_0 \times \{0, 1, 2, 3\}\).

\[
\begin{align*}
\text{Figure: State (people, cans) } & = (n, k) = (4, 2) \\
\end{align*}
\]

- Stochastic process \((X(t), Y(t): t \in [0, \infty))\), where \(X(t)\) describes the queue and \(Y(t)\) describes the environment.
- Customer arrival stream is Poisson with rate \(\lambda\).
- Service time is exponential with rate \(\mu\).
- Replenishment lead time is exponential with rate \(\nu\).
Construction of $Q$

Figure: Possible system changes from (people, cans) $= (X(t), Y(t)) = (2, 1)$

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$Krenzler, Daduna$ (Uni HH)  $Queues$ in $rnd.$ $environment$  $GPSD$ $2014$
Structure of the $Q$ matrices for $M/M/1/\infty$-queues with environment states $K$:

$$Q = \begin{pmatrix} B_0 & B_1 \\ A_{-1} & A_0 & A_1 \\ A_{-1} & A_0 & A_1 \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$B_i, A_i \in \mathbb{R}^{K \times K}$. 

Krenzler, Daduna (Uni HH)  Queues in rnd. environment  GPSD 2014  8 / 19
Solution of $\pi Q = 0$

$\lambda$ - arrival rate  
$\mu$ - service rate  
$\nu$ - replenishment rate

Figure: $(n, k) = (2, 1)$

For the limiting distribution $\pi(n, k) := \lim_{n \to \infty} P(X(t) = n, Y(t) = k)$ it holds

Product form!

$$\pi(n, k) = \xi(n) \theta(k)$$

with $\xi(n) = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n$ and $\theta = C_{\theta}^{-1} \left(\frac{\lambda}{\nu}, 1, \left(\frac{\lambda + \nu}{\lambda}\right), \left(\frac{\lambda + \nu}{\lambda}\right)\right)$.

Can we keep these properties of $\pi$ in more general settings?

YES, WE CAN!
## Loss system

<table>
<thead>
<tr>
<th></th>
<th>Vending machine</th>
<th>$\text{M/M/1/\infty}$-loss system</th>
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<tbody>
<tr>
<td><strong>arrival</strong></td>
<td>Poisson$(\lambda)$</td>
<td></td>
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<tr>
<td><strong>service, FCFS</strong></td>
<td>$\text{Exp}(\mu)$</td>
<td>&quot;$\text{Exp}(\mu(n))&quot;, X(t) = n$</td>
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<tr>
<td><strong>environment states</strong></td>
<td>$K = {0, 1, 2, 3}$</td>
<td>$K$ - countable</td>
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<tr>
<td><strong>env. states with no service and new customer loss</strong></td>
<td>${0}$ (empty machine)</td>
<td>$K_B \subset K$</td>
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<tr>
<td><strong>env. changes after service $n \geq 1$</strong></td>
<td>$(n, k) \rightarrow (n-1, k-1) = \mu, k \geq 1$</td>
<td>$(n, k) \rightarrow (n-1, m) = \mu R_{km}$, with stochastic matrix $R$</td>
</tr>
<tr>
<td><strong>env. changes independent from queue</strong></td>
<td>$(n, 1) \rightarrow (n, 3) = \nu$</td>
<td>$(n, k) \rightarrow (n, m) = V_{km}$, with generator matrix $V$</td>
</tr>
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$\nu$ = replenishment
Let \((X(t), Y(t))\) be an ergodic \(M/M/1/\infty\)-loss system with environment states \(K\) and system parameters: \(\lambda, \mu(n), K_B\) (resp. \(I_W\)), \(R, V\). Then for the limiting distribution it holds

\[
\pi(n, k) := \lim_{t \to \infty} P(X(t) = n, Y(t) = k) = \xi(n)\theta(k)
\]

with

\[
\xi(n) = C_\xi^{-1} \prod_{i=1}^{n} \left( \frac{\lambda}{\mu(i)} \right), \quad C_\xi - \text{normalization constant}
\]

and \(\theta\) the unique stochastic solution of

\[
\theta \lambda(I_W(R-I)+V) = 0 \quad \text{(easier to solve than } \pi Q = 0)\]

\(\in \mathbb{R}^{K \times K}\)
## Embedded Markov chains (EMC)

### What it is:
- Convert a continuous time process (CTP) into a Markov chain in discrete time.

### How?
- Observe the system right after customer leaves the queue.
- Calculate transition probabilities $P$.
- Solve $\hat{\pi}P = \hat{\pi}$.

### Why?
- "Classical method" to analyze $M/G/1/\infty$ queues, which are a superset of $M/M/1/\infty$ queues.
- Without environment the limiting distribution of $M/G/1/\infty$ modeled as EMC is the same as $M/G/1/\infty$ modeled as CTP.
Embedded Markov chains

\[ A_{1,3}^{(2,2)} = P((2, 1), (3, 2)) \]

\[ U((3, 1), \cdot) \quad R(\cdot, (3, 2)) \]

**Figure:** Probability to change from \((2, 1)\) to \((3, 2)\).

\[ U_{km}^{(i, n)} := P \left( (X(\tau_1), Y(\tau_1^-)) = (n+i-1, m) | Z(0) = (i, k) \right) \cdot A^{(i, n)} = U^{(i, n)} R \]
Embedded Markov chains

\[ B_{12}^{(2)} = P((0,1),(3,2)) \]

\[ W((0,1),\cdot) \quad U(\cdot,\cdot) \quad R(\cdot,(3,2)) \]

Figure: Probability to change from (0,1) to (3,2).

\[ W_{km} := P(Z(\sigma_1) = (1,m) | Z(0) = (1,k)) \]

\[ B^{(n)} = WU^{(1,n)}R \]
Transition probabilities

\[
P = \begin{pmatrix}
 WU^{(1,0)} R & WU^{(1,1)} R & WU^{(1,2)} R & WU^{(1,3)} R & \cdots \\
 U^{(1,0)} R & U^{(1,1)} R & U^{(1,2)} R & U^{(1,3)} R & \cdots \\
 0 & U^{(2,0)} R & U^{(2,1)} R & U^{(2,2)} R & \cdots \\
 0 & 0 & U^{(3,0)} R & U^{(3,1)} R & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

- before arrival:

\[
W = \lambda (\lambda l_W - V)^{-1} l_W
\]

- before departure:

\[
U^{(i,0)} = ((\lambda + \mu(i)) l_W - V)^{-1} \mu(i) l_W
\]

\[
U^{(i,n+1)} = U^{(i,n)} \left( \frac{\lambda}{\mu(n+i)} \right) \mu(n+1+i) (\lambda l_W + \mu(n+1+i) l_W - V)^{-1}
\]

Solve \( \hat{\pi} P = \hat{\pi} \)
Results

Let \((X(t), Y(t) : t \in \mathbb{R}_0)\) be an ergodic \(M/M/1/\infty\)-loss system with states \(K\) and system parameters: \(\lambda, \mu(n), K_B\) (resp. \(I_W\), \(R\), \(V\)). And let \((\hat{X}(t), \hat{Y}(t)) : t \in \mathbb{N}_0\) be the appropriate Markov chain. Then for the limiting distribution it holds

\[
\hat{\pi}(n, k) := \lim_{t \to \infty} P(\hat{X}(t) = n, \hat{Y}(t) = k) = \xi(n)\hat{\theta}(k)
\]

with

\[
\xi(n) = C_\xi^{-1} \prod_{i=1}^{\infty} \left(\frac{\lambda}{\mu(i)}\right)^n, \quad C_\xi - \text{normalization constant}
\]

and \(\hat{\theta}\) the unique stochastic solution of

\[
\hat{\theta} \left( I_W - \frac{1}{\lambda} V \right)^{-1} I_W R = \hat{\theta} \quad \text{(easier to solve than } \hat{\pi}P = \hat{\pi})
\]
### Continuous time vs embedded Markov chains

<table>
<thead>
<tr>
<th>Continuous time</th>
<th>Embedded Markov chains</th>
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<tbody>
<tr>
<td>transition rates $Q$</td>
<td>transition probabilities $P$</td>
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<tr>
<td>solve $\pi Q = 0$</td>
<td>solve $\hat{\pi} P = \hat{\pi}$</td>
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<tr>
<td>$\pi(n, k) = \xi(n) \theta(k)$</td>
<td>$\hat{\pi}(n, k) = \xi(n) \hat{\theta}(k)$</td>
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$$
\xi(n) = C_\xi^{-1} \prod_{i=1}^n \left( \frac{\lambda}{\mu(i)} \right)
$$

$$
\theta \lambda (I_W (R - I) + V) = 0 \quad \text{and} \quad \hat{\theta} C_{\hat{\theta}}^{-1} (I_W - \frac{1}{\lambda} V)^{-1} = \hat{\theta}
$$

in general $\pi \neq \hat{\pi}$ (different from just a queue)

$$
\theta = \left( \hat{\theta} (I_W - \frac{1}{\lambda} V)^{-1} e \right)^{-1} \hat{\theta} (I_W - \frac{1}{\lambda} V)^{-1}
$$

$$
\hat{\theta} = (\theta I_W e)^{-1} \cdot \theta I_W R
$$
Ongoing research

- Why no product form for non-constant arrival rate?
- Is exponential distribution necessary for the product form?
- Extend results to networks
- Link to similar problems: boundaries, starting points
Thank you for your attention!
$M/M/1/\infty$-loss system

Figure: Loss systems with parameters $\lambda$, $\mu(n)$, $K_B$ (resp. $I_W$), $R$, $V$. 
Relation between $I_W$ and $K_B$

The matrix $I_W \in \{0, 1\}^{K \times K}$ is a special way to write the blocking states $K_B$ in a matrix form.

$$(I_W)_{km} := \delta_{km}1_{[k \notin K_B]}$$

Example $K = \{0, 1, 2, 3\}$, $K_B = \{0\}$

$I_W = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 1
\end{pmatrix}$
Soft drink vending machine: $\theta$-solution

$$I_W = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -\nu & 0 & \nu \\ 1 & 0 & -\nu & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

$$\theta(\lambda I_W(R - I) + V) = 0$$

$$\theta(0) = \frac{\lambda}{\nu} \theta(1), \theta(2) = \frac{(\nu + \lambda)}{\lambda} \theta(1), \theta(2) = \theta(3)$$

Normalization: $C_{\theta} = \sum_{k=0}^{3} \theta(k) = \left(\frac{\lambda}{\nu} + \frac{2\nu}{\lambda} + 3\right) \theta(1) \implies \theta(1) = \frac{1}{\left(\frac{\lambda}{\nu} + \frac{2\nu}{\lambda} + 3\right)}$
The paths of $Z$ are cadlag. With $\tau_0 = \sigma_0 = \zeta_0 = 0$ and
\[
\tau_{n+1} := \inf(t > \tau_n : X(t) < X(\tau_n)), \quad n \in \mathbb{N}.
\]
denote the sequence of departure times of customers by $\tau = (\tau_0, \tau_1, \tau_2, \ldots)$, and with
\[
\sigma_{n+1} := \inf(t > \sigma_n : X(t) > X(\sigma_n)), \quad n \in \mathbb{N},
\]
denote by $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ the sequence of instants when arrivals are admitted to the system (because the environment is in states of $K_W$, i.e., not blocking)
and with
\[
\zeta_{n+1} := \inf(t > \zeta_n : Z(t) \neq Z(\zeta_n)), \quad n \in \mathbb{N},
\]
denote by $\zeta = (\zeta_0, \zeta_1, \zeta_2, \ldots)$ the sequence of jump times of $Z$. 
Matrix invertible?

\[ \hat{\theta} \left( \lambda \left( I_W - \frac{1}{\lambda} V \right) \right)^{-1} I_W R = \hat{\theta} \]

Properties of \( I_W - \frac{1}{\lambda} V \) (in general):
- "to some extent" diagonally dominant
- "to some extent" irreducible

Known facts for finite dimensional matrices:
- diagonally dominant \( \implies \) invertible
- irreducible weakly diagonally dominant \( \implies \) invertible

More general condition for irreversibility (finite dimensional)
Combine and extend: "to some extent" diagonally dominant and "to some extent" irreducible are sufficient.
Matrix Invertible?

Theorem

Let $M \in \mathbb{R}^{K \times K}$, where the set of indices is partitioned according to $K = K_W + K_B$, $K_W \neq \emptyset$, and $|K| < \infty$, whose diagonal elements have following properties:

\[
|M_{kk}| = \sum_{m \in K \setminus \{k\}} |M_{km}|, \quad \forall k \in K_B \tag{1}
\]

\[
|M_{kk}| > \sum_{m \in K \setminus \{k\}} |M_{km}|, \quad \forall k \in K_W \tag{2}
\]

and it holds the flow condition

\[
\forall \tilde{K}_B \subset K_B, \tilde{K}_B \neq \emptyset: \quad \exists \quad k \in \tilde{K}_B, \quad m \in \tilde{K}_B^c : \quad M_{km} \neq 0. \tag{3}
\]

Then $M$ is invertible.
Ruslan Krenzler and Hans Daduna.

*Loss systems in a random environment.*

December 2013.

http://arxiv.org/abs/1312.0539
23 March 2014: Corrected expression for the environment equation to
\[ \theta(\lambda I_{W}(R - I) + V) = 0 \]