BP as a multiplicative Thom spectrum

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Introduction

This diploma thesis investigates an attempt to show that the Brown-Peterson spectrum BP has a strictly commutative multiplication. More precisely, it proves that BP is not the localization of a Thom spectrum $M(f)$ associated to a second and thus an infinite loop map. If it was this would imply an E_∞-structure and hence a strictly commutative multiplication on BP.
The result is due to Priddy but is unfortunately unpublished. L. G. Lewis mentions it in his Ph. D. thesis and gives a short description of the proof (see [14], p.145). Its details will be explained here.
Cause of this thesis was a paper of Birgit Richter where she proves that BP cannot be a Thom spectrum associated to n-fold loop maps to $BS\mathcal{F}$ for certain $n > 2$ by use of Dyer-Lashof operations (see [23], section 7).

Spectra, as they will be presented here, are a sequence of topological spaces with additional structure. They play an important role in stable homotopy theory where we say a phenomenon is stable, if it occurs in any sufficiently large dimension and in essentially the same way independent of dimension (compare [2]).
In particular, the Brown-Peterson spectrum BP is of interest in the context of calculating stable homotopy groups of spheres. The best tool to calculate these groups is presently the Adams-Novikov spectral sequence with use of the Thom spectrum MU. Unfortunately, MU is rather difficult to handle since $\pi_*(MU)$ is a polynomial algebra with generators whose degree increases linearly. Instead, one considers localizations of MU and this is the great entrance of BP: The p-localization of MU splits into copies of BP and fortunately, $\pi_*(BP)$ is a polynomial algebra with generators whose degree increases exponentially.
Knowing that BP is of relevance when considering such an important problem as stable homotopy groups of spheres, one naturally asks what kind of structure there is on BP. Definitely, one would like to have a commutative multiplication on it. However, when it comes to spectra, multiplication very often is only homotopy-commutative. Nevertheless, there are theorems giving conditions for a Thom spectrum to be strictly commutative and saying that the localization of a spectrum inherits the striclty commutative structure. Thus, since BP arises in the p-localization of the Thom spectrum MU it is obvious to try to understand

BP as the localization of a Thom spectrum satisfying the conditions of these theorems and thereby prove that BP is strictly commutative. However, this attempt fails and this thesis will explain why.

The most important chapter - besides the last one of course - is the one about spectra. The theory of spectra forms the background of the question that will be discussed. However, the theory of spectra is very complex. There are different models of spectra and the first encounter might be a little bit confusing. For a better understanding, I tried to present the required facts as homogenous as possible. Disappointingly, we need some deep theorems I can only state but not give any details as this would require too much theory.
The other chapters essentially deliver technical tools needed for the proof. Mainly, these tools are the Eilenberg-Moore spectral sequence (chapter (2)) and secondary cohomology operations (chapter (4)).
Finally, in chapter (5) I prove that BP is not the localization of a Thom spectrum $M(f)$ associated to a two-fold loop map following the outline in [14] and filling in the details.

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Contents

Chapter 1

Spectra

There are several models of spectra. We will give a definition in the sense of a sequential spectrum because this bears two advantages: Firstly, it is the easiest approach to the complex theory of spectra and secondly, there are nice sequential models of the spectra we need. Naturally, there are also disadvantages which will be explained later. Our notation and nomenclature follows mainly Adams' presentation in [2] and Switzer's in [25].

Definition 1.0.1 *A spectrum E is a sequence of pointed topological spaces E_n together with basepoint-preserving maps $\Sigma E_n = \mathbb{S} \wedge E_n \to E_{n+1}$.*

Definition 1.0.2 *We call a spectrum E a CW-spectrum if E_n is a CW-complex with basepoint for all n and each map $\Sigma E_n \to E_{n+1}$ is an homeomorphism from ΣE_n to a subcomplex of E_{n+1}.*

Example 1.0.3 *An easy but important example is the sphere spectrum S with $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$. It is a 'natural' case of the suspension spectrum $\Sigma^\infty X$ where we start with an arbitrary pointed space X and define $E_n = \Sigma^n X$.*

Example 1.0.4 *A further class of examples are Ω-spectra. A Ω-spectrum is a sequence of CW complexes $E_1, E_2, \dots$ together with homotopy equivalences $E_n \to \Omega E_{n+1}$ for all n. This is a spectrum in the above sense because of the adjoint relation $[\Sigma X, Y] = [X, \Omega Y]$.*

Example 1.0.5 *Let E be a spectrum and X a CW complex. Then $E \wedge X$ is a spectrum with $(E \wedge X)_n = E_n \wedge X$ and the obvious maps $\Sigma(E_n \wedge X) \cong \Sigma E_n \wedge X \to E_{n+1} \wedge X$.*

The spectrum defined in the last example will be of some importance as it will be needed for the definition of the E-homology of a CW-complex X.

1.1 Maps between spectra

An obvious attempt of defining a map between spectra would be to define it on space level and demand that it commutes with the maps $\Sigma E_n \to E_{n+1}$. However, this would be too strict and there are cases in which we do not find enough maps like this to do what we want (see for example [2], pp.141,142). Thus, we will call them function and then present a more appropriate notion of a map between spectra.

Definition 1.1.1 *A function $f : E \to F$ between spectra (of degree 0) is a sequence of maps $f_n : E_n \to F_n$ such that the following diagram is strictly commutative for each n*

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_n \arrow[r] & F_{n+1}. \end{tikzcd}$$

Definition 1.1.2 *Let E be a CW-spectrum. A supspectrum $E' \subset E$ is called cofinal if for each n and each finite subcomplex $K \subset E_n$ there is an m such that $\Sigma^m K$ maps into E'_{m+n}. That is, each cell in each E_n is sent to E' after enough suspensions.*

With this we can now actually define a map between spectra.

Definition 1.1.3 *Let E be a CW-spectrum and F a spectrum. Take all cofinal subspectra $E' \subset E$ and all functions $f' : E' \to F$. We say that two functions $f' : E' \subset F$ and $f'' : E'' \to F$ are equivalent if there is a cofinal subspectrum E''' contained in E' and E'' such that the restrictions of f' and f'' to E''' coincide. We call an equivalence class of such functions a map from E to F and it is represented by a pair (E', f').*

Naturally, we want to compose maps. Let E, F be CW-spectra and G a spectrum. Then define the composition of maps $E \to F, F \to G$ by composition of representatives. Obviously, for this purpose we need to know that for each function $E \to F$ and a cofinal subspectrum $F' \subset F$ there exists a cofinal subspectrum $E' \subset E$ such that E' is mapped into F'. Moreover, we need that if E' is a cofinal subspectrum of E and E'' is a cofinal subspectrum of E', then E'' is in fact a cofinal subspectrum of E. Both statements are of course true (see [2], p.143).

Finally, we explain what we mean by homotopic maps of spectra.

Definition 1.1.4 *Let I^+ be the union of the unit intervall and a disjoint basepoint. Two maps of spectra $f_0, f_1 : E \to F$ are homotopic if there is a homotopy*

$h : E \wedge I^+ \to F$ with $h \circ i_0 = f_0, h \circ i_1 = f_1$, whereas $i_0, i_1 : E \to E \wedge I^+$ are the maps induced by the inclusions of 0 and 1 into I^+. We write $[E, F]$ for the set of homotopy classes of maps from E to F.

In the following, we will work in the stable homotopy category of spectra as Adams defines it. That is, the objects of our category are CW-spectra and its morphisms are homotopy classes of maps. This restriction is not too strict, since every spectrum is weakly equivalent to a CW-spectrum (see [2], p.157, for example).

1.2 Smash product of spectra

In order to explain a multiplicative structure on a spectrum E we need a smash product $E \wedge E$. However, the construction of the smash product of two CW-spectra is rather complicated. We will therefore only present the idea of the construction.

As a first attempt, we would want $E \wedge F$ to be thing to which $E_n \wedge F_m$ tends as n and m go to infinity. This idea leads to the following construction of the now called *naive smash product* which goes back to J. M. Boardman.

Let A be some ordered set isomorphic to $\mathbb{N}$ as ordered set and let $B \subset A$ be a subset. We define a monotonic function $\beta : A \to \mathbb{N}$ by saying that $\beta(a)$ is the number of elements $b \in B$ with $b < a$. In particular we have $\alpha : A \to \mathbb{N}$ corresponding to $A \subset A$. We then suppose that A is the union of two subsets B, C with $B \cap C = \emptyset$ and β, γ being the corresponding functions. Evidently, $\beta(a) + \gamma(a) = \alpha(a)$ for all $a \in A$. We define the naive smash product $E \wedge_{BC} F$ by

$$(E \wedge_{BC} F)_{\alpha(a)} = E_{\beta(a)} \wedge F_{\gamma(a)}.$$

In order to define the maps of this product spectrum, we regard $\mathbb{S}^1$ as $\mathbb{R}^1$ compactified by adding a point at infinity, which becomes the base point. This allows us to define a map of degree -1 from $\mathbb{S}^1$ to $\mathbb{S}^1$ by $t \mapsto -t$.
Let $e \in E_{\beta(a)}, f \in F_{\gamma(a)}, t \in \mathbb{S}^1$ and $\zeta : \mathbb{S}^1 \wedge E_{\beta(a)} \to E_{\beta(a)+1}, \eta : \mathbb{S}^1 \wedge F_{\gamma(a)} \to F_{\gamma(a)+1}$ be the appropriate maps from E and F.
If $a \in B$ then $(E \wedge_{BC} F)_{\alpha(a)+1} = E_{\beta(a)+1} \wedge F_{\gamma(a)}$ and we define

$$\pi_{\alpha(a)} : \mathbb{S} \wedge (E \wedge_{BC} F)_{\alpha(a)} \to (E \wedge_{BC} F)_{\alpha(a)+1}$$

by $\pi_{\alpha(a)}(t \wedge e \wedge f) = \zeta_{\beta(a)}(t \wedge e) \wedge f$.
If $a \in C$ then $(E \wedge_{BC} F)_{\alpha(a)+1} = E_{\beta(a)} \wedge F_{\gamma(a)+1}$ and $\pi_{\alpha(a)}$ is defined by $\pi_{\alpha(a)}(t \wedge e \wedge f) = e \wedge \eta_{\gamma(a)}((-1)^{\beta(a)} t \wedge f)$.

The smash product we have so far constructed is natural with respect to functions, but if B or C is finite, we will get problems with maps. Moreover, we get a naive smash product $E \wedge_{BC} F$ for each partition $A = B \cup C$, some of them being commutative, some associative. So which partition shall we choose if we want our smash product to have all these nice properties? In fact, we will not choose a particular partition, but pick all the possible spectra $E \wedge_{BC} F$ and put them together in a construction called *telescope*.

Definition 1.2.1 *Given a sequence $\mathcal{X} = \{\cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \ldots\}$ of maps, we define its telescope $T\mathcal{X}$ to be the space*

$$T\mathcal{X} := \left(\bigcup (X_n \times [n, n+1])\right) / \sim$$

where $(x, n+1) \in X_n \times [n, n+1] \sim (f_n(x), n+1) \in X_{n+1} \times [n+1, n+2]$.

To define the telescope TE of a spectrum E, let us regard again $\mathbb{S}^n$ as $\mathbb{R}^n$ compactified by adding a point at infinity that becomes the basepoint. The isomorphism $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{m+n}$ then gives an isomorphism $\mathbb{S}^n \wedge \mathbb{S}^m \cong \mathbb{S}^{m+n}$ which should be kept in mind during the following definition. Let

$$TE_n = \left((\bigvee_{i=0}^{n} \mathbb{S}^{n-i} \wedge E_i \wedge [i]^+) \vee (\bigvee_{i=0}^{n-1} \mathbb{S}^{n-i} \wedge E_i \wedge [i, i+1]^+) \right) / \sim$$

where

$$[t, e, i] \in \mathbb{S}^{n-i} \wedge E_i \wedge [i]^+ \sim [t, e, i] \in \mathbb{S}^{n-i} \wedge E_i \wedge [i, i+1]^+$$

and

$$[t, \zeta(s \wedge e), (i+1)] \in \mathbb{S}^{n-i-1} \wedge E_{i+1} \wedge [i+1]^+ \sim [s, t, e, (i+1)] \in \mathbb{S}^1 \wedge \mathbb{S}^{n-i-1} \wedge E_i \wedge [i, i+1]^+.$$

Thereby, $\zeta : \mathbb{S}^1 \wedge E_i \to E_{i+1}$.

Finally, one can construct $E \wedge F$ as a kind of 'two-dimensional telescope'. This construction is similar to the one above, though more complicated and longer. The interested reader may take a look at [2], $(III.4)$. As promised, this smash product stands in close relation to the naive smash product discussed before.

Lemma 1.2.2 *$eq_{BC} : E \wedge_{BC} F \to E \wedge F$ is a homotopy equivalence if any one of the following is satisfied:*

1. *B and C are both infinite*
2. *B has d elements and $\Sigma E_r \cong E_{r+1}$ for $r \geq d$*
3. *C has d elements and $\Sigma F_r \cong F_{r+1}$ for $r \geq d$*

The smash product $E \wedge F$ has the following properties:

Theorem 1.2.3 (a) *$X \wedge Y$ is a covariant functor of two variables with arguments and values in our stable homotopy category.*
(b) *There are natural homotopy equivalences (natural in the above category)*

$$\begin{aligned} a = a(E,F,G) &: (E \wedge F) \wedge G \to E \wedge (F \wedge G) \\ \tau = \tau(E,F) &: E \wedge F \to F \wedge E \\ l = l(E) &: S \wedge E \to E \\ r = r(E) &: E \wedge S \to E \end{aligned}$$

with S being the sphere spectrum.

A proof may be found in [2], III.4, or [25], chapter 13.

The construction of the smash product we presented so far is rather long and intricate. In addition, we constructed it in our stable homotopy category where we only have homotopy classes maps. Naturally, we would like to have a product on a category of spectra where we have maps as morphisms. Unfortunately, we do not have this for sequential spectra. However, as we said in the beginning, there are other models of spectra with more structure. Some of them do have a strictly commutative (smash) product.

1.3 Spectra and (Co-)homology theories

What makes spectra so special is that one gets a (co-)homology theory out of each spectrum, and each (co-)homology theory can be represented by a spectrum.

Definition 1.3.1 *We define the homotopy groups of a spectrum E to be $\pi_i(E) = \varinjlim \pi_{i+n}(E_n)$ where the direct limit is computed using the composition*

$$\pi_{i+n}(E_n) \to \pi_{i+n+1}(\Sigma E_n) \to \pi_{i+n+1}(E_{n+1}).$$

In the case of the suspension spectrum of a space X, the homotopy groups of the spectrum are the same as the stable homotopy groups of X.

Proposition 1.3.2 *Let E be a CW-spectrum and X a CW-complex. The groups*

$$E_n(X) = \pi_n(E \wedge X) = [\mathbb{S}^n, E \wedge X]$$

form a reduced homology theory, called E-homology of X. Moreover, the groups

$$E^n(X) = [\Sigma^\infty X, \Sigma^n E],$$

where Σ^∞ is the suspension spectrum of X defined in example (1.0.3), form a reduced cohomology theory, called E-cohomology of X. For $f : (X, x_0) \to (Y, y_0)$ we take $E_n(f) = (1 \wedge f)_*$ and $E^n(f) = (\Sigma^\infty f)^*$ respectively. Both theories satisfy the wedge axiom.

An explicit proof can be found in [25]. We want to concentrate on the extension of this definition on a category of spectra.

Definition 1.3.3 *Let $f : E \to F$ be a map of CW-spectra. To begin with, we define $CE := E \wedge I$ where I is given the basepoint 0. We then define the mapping cone $F \cup_f CE$ as the spectrum with $(F \cup_f CE)_n = F_n \cup_{f'_n} (E'_n \wedge I)$ where (E', f') represents f.*

Remark: $F \cup_f CE$ is well-defined. If (E'', f'') is another representative, then the mutual cofinal supspectrum E''' of E' and E'' extends to a mutual cofinal subspectrum $\{F_n \cup_{f'''} (E'''_n \wedge I)\}$ of $\{F_n \cup_{f'} (E'_n \wedge I)\}$ and $\{F_n \cup_{f''} (E''_n \wedge I)\}$. Thus, the last two are equivalent.

Definition 1.3.4 *For any map of CW-spectra $f : E \to F$ we call the sequence $(*) : E \xrightarrow{f} F \xrightarrow{j} F \cup_f CE$ and any sequence equivalent to it a cofiber sequence. A sequence that is equivalent to $(*)$ is a sequence $G \xrightarrow{g} H \xrightarrow{h} K$ for which there is a homotopy commutative diagram*

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "\alpha"] & H \arrow[r, "h"] \arrow[d, "\beta"] & K \arrow[d, "\gamma"] \\
E \arrow[r, "f"] & F \arrow[r, "j"] & F \cup_f CE
\end{tikzcd}$$

with α, β, γ being homotopy equivalences.

Proposition 1.3.5 *Let E, F be CW-spectra. Then*

$$E_n(F) = [\mathbb{S}^n, E \wedge F] \text{ and } E^n(F) = [F, \Sigma^n E] \text{ respectively}$$

form a homology theory and cohomology theory respectively in the following sense:
(1) $E_*(F)$ is a covariant functor of two variables E, F and with values in the category of (abelian) groups. $E^*(F)$ is a functor between the same categories which is covariant in E and contravariant in F.
(2) If $F \xrightarrow{f} G \xrightarrow{g} H$ is a cofiber sequence (of CW-spectra) and E is a CW-spectrum, then

$$E_n(F) \xrightarrow{f_*} E_n(F) \xrightarrow{g_*} E_n(H) \text{ and } E^n(F) \xleftarrow{f^*} E^n(F) \xleftarrow{g^*} E^n(H)$$

are exact.
(3) There are natural isomorphisms $E_n(F) \cong E_{n+1}(\Sigma F) = E_{n+1}(\mathbb{S}^1 \wedge F)$, $E^n(F) \cong E^{n+1}(\Sigma F)$.

Remark: Statement (2) is equivalent to the usual exactness axiom (compare definition (6.1.1)) of an reduced homology theory.

Sketch of the proof: (Compare [2].)
(1) Follows by definition.
(2) We restrict to here to the homology case since this is the one that requires a little bit of work. For the cohomology case see [2], p. 155.

Clearly, if we show that $[W, E] \xrightarrow{f_*} [W, F] \xrightarrow{i_*} [W, F \cup_f CE]$ is exact for $E \xrightarrow{f} F \xrightarrow{i} F \cup_f CE$ then the homology case will be a corollary of this.
Let g be an element in $[W, F]$ such that $i \circ g \simeq 0$. We have to show that $g \simeq f \circ l$ for some $l \in [W, E]$. To see this, consider the following diagram:

$$\begin{tikzcd}
E \arrow[r, "f"] & F \arrow[r, "i"] & F \cup_f CE \arrow[r] & \Sigma E \arrow[r, "-\Sigma f"] & \Sigma F \\
W \arrow[r] & W \arrow[u, "g"] \arrow[r] & CW \arrow[u, "h"] \arrow[r] & \Sigma W \arrow[u, "k"] \arrow[r, "-\mathrm{id}"] & \Sigma W \arrow[u, "\Sigma g"']
\end{tikzcd}$$

The maps in the lower row are the obvious ones and the map h exists because of $i \circ g \simeq 0$. The only non-obvious map is definitely $F \cup_f CE \to \Sigma E$. This follows from the fact that we can extend the cofibre sequence $E \xrightarrow{f} F \xrightarrow{i} F \cup_f CE$ to the right by adding another mapping cone:

$$E \xrightarrow{f} F \xrightarrow{i} F \cup_f CE \longrightarrow (F \cup_f CE) \cup_i CF \, .$$

Moreover, we have to know that this last spectrum is equivalent to $Y \cup_f CX)/Y$ and that this one is in fact ΣX.
Recall now that we wanted to show that $g \simeq f \circ l$ for some $l \in [W, E]$. The map $k : \Sigma W \to \Sigma X$ comes from a map $l \in [W, X] : k = \Sigma l$. Then the last square on the right tells us that $\Sigma(f \circ l) \simeq \Sigma g$ and thus $g = f \circ l$.
(3) The cohomology case is obvious. In the case of homology we need that $X \to \mathbb{S}^1 \wedge X$ is an equivalence of degree one (see for example [2]). This gives $E_n(X) = [\mathbb{S}^n, E \wedge X] \cong [\mathbb{S}^{n+1}, E \wedge \mathbb{S}^1 \wedge X] = E_{n+1}(\Sigma X)$. □

Example 1.3.6 (Eilenberg-MacLane spectrum) *Let G be an abelian group. Then the Eilenberg-MacLane spectrum HG is a Ω-spectrum with spaces $HG_n = K(G, n)$ and maps $K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1)$.*
It is

$$\pi_n(HG) = \begin{cases} G & n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

and the corresponding homology theory on a CW-complex is ordinary singular homology: $HG_i(X) \cong H_i(X; G)$.

We see here that we get a homology and cohomology theory respectively out of each spectrum. So what about the way back? Do we find for each (co-)homology theory a spectrum that represents it? Satisfactorily, the answer is 'yes' though this way is not that easy. In the case of cohomology, it follows from Brown's representability theorem which we will present in the following. In the case of homology however, things are again a little bit more complicated. (See for example [2], pp. 199,200)

Definition 1.3.7 *A contravariant functor F on $\mathcal{PW}'$, the category of pointed CW-complexes and homotopy classes of basepoint preserving maps, fulfills the Mayer-Vietoris Axiom, if for any CW-triad $(X; A_1, A_2)$, that is $X = A_1 \cup A_2$, and for any $x_1 \in F(A_1), x_2 \in F(A_2)$ with*

$$i_1^*(x_1) = i_2^*(x_2) \in F(A_1 \cap A_2), i_j : A_1 \cap A_2 \to A_j, j = 1, 2,$$

there is a $y \in F(X)$ with

$$i_1^{'*}(y) = x_1 \in F(A_1), i_2^{'*}(y) = x_2 \in F(A_2), i_j^{'} : A_j \to X, j = 1, 2.$$

Definition 1.3.8 *Let $\mathcal{PS}$ be the category of pointed sets and functions preserving basepoints and let $F : \mathcal{PW}' \to \mathcal{PS}$ be a contravariant functor satisfying the Wedge axiom (defined in* (6.1.2)*) and the Mayer-Vietoris axiom. An element $u \in F(Y)$ is called n-universal if*

$$T_u : [S^q, s_0; Y, y_0] \to F(S^q)$$

is an isomorphism for $q < n$ and an epimorphism for $q = n$. We call u universal if it is n-universal for all $n \geq 0$.

Theorem 1.3.9 (Brown's theorem) *If $F : \mathcal{PW}' \to \mathcal{PS}$ is a contravariant functor as above, then there is a classifying space $(Y, y_0) \in \mathcal{PW}'$ and an universal element $u \in F(Y)$ such that $T_u : [-; Y, y_0] \to F$, $T_u[f] = f^*(u) \in F(X)$ for any $f : (X, x_0) \to (Y, y_0)$, is a natural equivalence.*

A proof may be found in [25], chapter 9, for example. The main work lies in constructing appropriate (Y, y_0) and $u \in F(Y)$.

As a special case, we get the analogon to example (1.3.6).

Theorem 1.3.10 *Let X be a CW-complex. There are natural bijections $T : [X, K(G, n)] \to H^n(X; G)$ for all $n > 0$ with G being any abelian group. Such T is given as follows: Let α be a certain distinguished class in $H^n(K(G, n); G)$. Then for each class $x \in H^n(X; G)$, there exists a map $f : X \to K(G, n)$, unique up to homotopy, such that $f^*(\alpha) = x$. Thus, $T([f]) = f^*(\alpha)$.*

Such a class $\alpha \in H^n(K(G, n); G)$ is called a fundamental class. The proof of the theorem (see [9], Section 4.3) yields an explicit fundamental class, namely the element of $H^n(K(G, n); G) \cong Hom(H_n(K(G, n); \mathbb{Z}), G)$ given by the inverse of the Hurewicz isomorphism $G = \pi_n(K(G, n)) \to H_n(K(G, n); \mathbb{Z})$.

1.4 Ringspectra

Definition 1.4.1 *A ring spectrum is a CW-spectrum E with product $\mu : E \wedge E \to$ and unit $\iota : \mathbb{S}^0 \to E$ such that the following diagrams commute up to homotopy:*

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu \wedge id_E"] \arrow[d, "id_E \wedge \mu"] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"] & E \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^0 \wedge E \arrow[r, "\iota \wedge id_E"] \arrow[dr] & E \wedge E \arrow[d, "\mu"] & E \wedge \mathbb{S}^0 \arrow[l, "id_E \wedge \iota"'] \arrow[dl] \\ & E & \end{tikzcd}$$

μ is homotopy-commutative if the diagram

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

also commutes up to homotopy.

We see here that we so far only get commutativity up to homotopy. Naturally, we are also interested in strict commutativity. In the case of sequential spectra, the smash product is commutative only up to homotopy, so how could the above diagram be strictly commutative? Moreover, in the case of a different model of spectra which has a strictly commutative smash product and thus a chance to have a strictly commutative multiplication this turns out very hard to prove. In most cases, it is in fact easier to prove that there is an action of an operad on the spectrum in question that then induces commutativity. Therefore, we will now introduce operads and explain shortly how they induce a commutative multiplicative structure on a spectrum.

Definition 1.4.2 *An operad $\mathcal{O}$ is a collection of spaces $\{\mathcal{O}(k)\}_{k\geq 0}$ together with an element $1 \in \mathcal{O}(1)$ and maps*

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \to \mathcal{O}(j_1 + \cdots + j_k)$$

for each choice of $k, j_1, \ldots, j_k \geq 0$ such that
(a) for each k and each $s \in \mathcal{O}(k)$, $\gamma(1, s) = s$ and $\gamma(s, 1, \ldots, 1) = s$, and

(b) the following diagram commutes for all choices of $k, j_1, \ldots, j_k, i_{11}, \ldots, i_{k,j_k}$*:*

$$\begin{tikzcd}
\mathcal{O}(k) \times \prod_{m=1}^{k} \left(\mathcal{O}(j_m) \times \prod_{n=1}^{j_m} \mathcal{O}(i_{mn}) \right) \arrow[r, "1\times\gamma"] \arrow[d, "="'] & \mathcal{O}(k) \times \prod_{m=1}^{k} \mathcal{O}(i_{m1} + \cdots + i_{mj_m}) \arrow[dd, "\gamma"] \\
\left(\mathcal{O}(k) \times \prod_{m=1}^{k} \mathcal{O}(j_m) \right) \times \prod_{m,n} \mathcal{O}(i_{mn}) \arrow[d, "\gamma\times 1"'] & \\
\mathcal{O}(j_1 + \cdots + j_k) \times \mathcal{O}(i_{11} \times \cdots \times \mathcal{O}(i_{kj_k}) \arrow[r, "\gamma"] & \mathcal{O}(i_{11} + \cdots + i_{kj_k})
\end{tikzcd}$$

Moreover, if S_k *is the symmetric group of* k *elements, for each* k *there is a right action* ρ *of* S_k *such that for each* $\sigma \in S_k$ *and* $\tau_i \in S_{j_i}$ *the following diagrams commute:*

$$\begin{tikzcd}
\mathcal{O}(k) \times \prod \mathcal{O}(j_i) \arrow[r, "\gamma_{k,j_1,\ldots,j_k}"] & \mathcal{O}(j) & \mathcal{O}(j) \arrow[l, "\rho(\sigma)"'] \\
\mathcal{O}(k) \times \prod \mathcal{O}(j_i) \arrow[u, "\sigma\times 1"] \arrow[rr] & & \mathcal{O}(k) \times \prod \mathcal{O}(j_{\sigma^{-1}(i)}) \arrow[u, "\gamma_{k,j_{\sigma^{-1}(1)},\ldots,j_{\sigma^{-1}(k)}}"']
\end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd}
\mathcal{O}(k) \times \prod \mathcal{O}(j_i) \arrow[r, "\gamma"] & \mathcal{O}(j) \\
\mathcal{O}(k) \times \prod \mathcal{O}(j_i) \arrow[u, "1\times\prod\tau_i"] \arrow[r, "\gamma"] & \mathcal{O}(j) \arrow[u, "\tau_1\oplus\cdots\oplus\tau_k"']
\end{tikzcd} \tag{1.2}$$

Definition 1.4.3 *Let* $\mathcal{O}$ *be a operad and let* Y *be a space. An action of* $\mathcal{O}$ *on* Y *consists of a map* $\theta : \mathcal{O}(k) \times Y^k \to Y$ *for each* $k \geq 0$ *such that*
(a) $\theta(1, y) = y$ *for all* $y \in Y$*,*
(b) the following diagram commutes for all $k, j_1, \ldots, j_k \geq 0$*:*

$$\begin{tikzcd}
\mathcal{O}(k) \times \left(\prod_{m=1}^{k} \mathcal{O}(j_m) \right) \times Y^{j_1+\cdots+j_k} \arrow[r, "="] \arrow[dd, "\gamma\times 1"'] & \mathcal{O}(k) \times \prod_{m=1}^{k} (\mathcal{O}(j_m) \times Y^{j_m}) \arrow[d, "1\times\prod\theta"] \\
& \mathcal{O}(k) \times Y^k \arrow[d, "\theta"] \\
\mathcal{O}(j_1 + \cdots + j_k) \times Y^{j_1+\cdots+j_k} \arrow[r, "\theta"] & Y
\end{tikzcd}$$

(c) and $\theta : \mathcal{O}(k) \times Y^k \to Y$ *factors through* $\mathcal{O}(k) \times_{S_k} Y^k = \mathcal{O}(k) \times Y^k / \sim$*, with* $(\tau(c), y) \sim (c, \tau(y))$ *for* $\tau \in S_k, c \in \mathcal{O}(k), y \in Y^k$ *and* S_k *acts on* Y^k *in the obvious way.*

Example 1.4.4 *For any space Y, the collection $\{Map(Y^k, Y)\}_{k\geq 0}$ together with the compact-open topology and its usual multivariable composition builds an operad, called the endomorphism operad, and we will denote it End(Y). Obviously, End(Y) acts on Y in the described way.*

Definition 1.4.5 *An E_∞ operad is an operad $\mathcal{O}$ for which each space $\mathcal{O}(k)$ is weakly equivalent to a point.*

Remark: There are other definitions of an E_∞ operad (e.g. $\mathcal{O}$ is E_∞ if S_k acts freely on $\mathcal{O}(k)$), but this is the most useful one for us here.

An important fact about E_∞-spaces is the following one:

Theorem 1.4.6 *Let Y be a topological space. Then Y is weakly equivalent to an infinite loop space if and only if Y has a grouplike action of an E_∞ operad.*

(See [20].)

Moreover, a space with an action of an E_∞ operad inherits structure from this action. In particular, the action of $\mathcal{O}(2)$ induces a multiplication on Y: $\mathcal{O}(2) \times Y^2 \to Y, (c, y_1, y_2) \mapsto \mu(y_1, y_2)$. This multiplication is unique up to homotopy, since $\mathcal{O}(2)$ is weakly equivalent to a point and thus path-connected. In fact, it is homotopy-associative and -commutative.

Associativity:
Consider

$$\mathcal{O}(2) \times \mathcal{O}(2) \times \mathcal{O}(1) \times Y^3 \xrightarrow{\gamma} \mathcal{O}(3) \times Y^3 \xrightarrow{\theta} Y,$$
$$(c, c, 1, y_1, y_2, y_3) \longmapsto \mu(\mu(y_1, y_2)y_3)$$

and

$$\mathcal{O}(2) \times \mathcal{O}(1) \times \mathcal{O}(2) \times Y^3 \xrightarrow{\gamma} \mathcal{O}(3) \times Y^3 \xrightarrow{\theta} Y,$$
$$(c, 1, c, y_1, y_2, y_3) \longmapsto \mu(y_1, \mu(y_2, y_3)).$$

Since $\mathcal{O}(3)$ is weakly equivalent to a point, it is path-connected. That is, $\gamma(c, c, 1)$ and $\gamma(c, 1, c)$ are homotopic and thus, $\mu(\mu\times 1)$ and $\mu(1\times\mu)$ are homotopic as well.

Commutativity:
Follows immediately from Definition (1.4.3), (c), if $\mathcal{O}(2)$ is path-connected.

The action of an operad on a spectrum E can be explained in the same way as for a space, if E^k denotes the k-fold smash product. However, we get again commutativity only up to homotopy. Fortunately, there is a theorem telling us that an E_∞-spectrum has a strictly commutative multiplication which may be found in [6]. Obviously, this does not make sense for sequential spectra. We definitely need other models of spectra here, for example symmetric spectra (see [12]) or spectra as 'S-modules' like they are explained in (see [6]).

1.5 Localization of a spectrum

1.5.1 Localization via Moore spectra

Theorem 1.5.1 (Moore spectrum) *For every abelian group A, there exists a spectrum $M(A)$ with the following properties:*
$i)\pi_i(M(A)) = 0, i < 0$
$ii)\pi_0(M(A)) = A = H_0(M(A))$
$iii)H_i(M(A)) = 0, i \neq 0$
$M(A)$ is called Moore-spectrum and is unique up to equivalence.

Definition 1.5.2 *Let A be a subring of $\mathbb{Q}$ and $M(A)$ the corresponding Moore-spectrum. We define the A-localization of E by $E_A := E \wedge M(A)$ for every spectrum E.*
Especially: $E_{(p)} = E_{\mathbb{Z}_{(p)}}$ with $\mathbb{Z}_{(p)} = \{\frac{m}{n} | m, n \in \mathbb{Z}, p \not| n\} \subset \mathbb{Q}$ denoting the p-local integers.

The following proposition states that E_A does indeed behave like a localization.

Proposition 1.5.3 *Let E be a spectrum and let A be a subring of the rational numbers $\mathbb{Q}$. Then there is a map of spectra $j : E \to E_A$ and for every spectrum X there is an isomorphism*

$$E_{A*}(X) \cong E_*(X) \otimes A$$

such that $j_* : E_*(X) \to E_{A*}(X)$ is given by $j_*(x) = x \otimes 1$.
Moreover, if E is a ringspectrum then ER is a ringspectrum and $J : E \to ER$ is a map of ringspectra.

The proof can be found in [13], pp. 168,169.

It will be of some importance if there is a strictly commutative ring structure on $E_{(p)}$ provided there is one on E. Luckily, the answer is yes. However, we have to make a detour in order to see this. There is another way to define a localization of spectra, the Bousfield localization. It may be regarded as a generalization of the localization discussed above since in certain cases these localizations are equal. Moreover, Bousfield localizations of strictly commutative ringspectra are again strictly commutative which makes this detour worth while.

1.5.2 Bousfield localization

Let E_* be a homology theory and E its representing spectrum.

Definition 1.5.4 *We call a spectrum F E_*-acyclic if $E_*(F) = 0$ and a spectrum G E_*-local if $[F, G] = 0$ for each E_*-acyclic spectrum F.*

Definition 1.5.5 *A map $f : F \to G$ of spectra is an E_*-equivalence if it induces an isomorphism in E_*-homology.*

Definition 1.5.6 *An E_*-localization functor is a covariant functor on the stable homotopy category $L_E : \mathcal{ST} \to \mathcal{ST}$ together with a natural transformation η from the identity functor to L_E such that $\eta_F : F \to L_E(F)$ is the terminal E_*-equivalence from F, that is η_F is an E_*-equivalence and for any E_*-equivalence $f : F \to G$ there is a unique $r : G \to L_E(F)$ such that $rf = \eta_F$:*

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[dr, "\eta"'] & G \arrow[d, "\exists! r"] \\ & L_E(F) \end{tikzcd}$$

Bousfield proved in [5] that there is a localization functor $L_E : \mathcal{ST} \to \mathcal{ST}$ for every spectrum E.

Definition 1.5.7 *We call a spectrum E n-connected if $\pi_i(E) = 0$ for $i \leq n$ and connected if it is (-1)-connected.*

The following proposition is the first step to see that $E_{(p)}$ is a strictly commutative ringspectrum if E is one:

Proposition 1.5.8 *Let E, F be connective spectra and E_* the homology theory corresponding to E such that at least one element of E_* has infinite order. Moreover, let J be a set of primes such that for each i E_i is uniquely p-divisible for each $p \notin J$. $\mathbb{Z}_{(J)} = \{\frac{m}{n} | m, n \in \mathbb{Z}, p \not| n \forall p \in J\}$ is the localization of $\mathbb{Z}$ on J. Then $L_E(F) \simeq M(\mathbb{Z}_{(J)}) \wedge F$.*

The proof can be found in [5].

Example 1.5.9 *The p-localization $M(\mathbb{Z}_{(p)}) \wedge E$ corresponds to the $H\mathbb{Z}_{(p)*}$-localization with $H\mathbb{Z}_{(p)}$ being the Eilenberg-MacLane spectrum for $\mathbb{Z}_{(p)}$.*

The second step is knowing that Bousfield localizations of E_∞-ringspectra are again E_∞. This follows from theorem 2.2 in [6], chapter VIII.

Chapter 2

Spectral sequences

The ideas presented in this chapter are mainly taken from a so far unfinished book of Hatcher whose first chapters are available on the internet (see [10]).

Definition 2.0.10 *(a) Let R be a ring. A differential bigraded R-module of homological and cohomological type respectively is a collection of R-modules $\{E^{pq}\}$ or $\{E_{pq}\}, p, q \in \mathbb{Z}$, together with an R-linear map $d_r : E^{pq} \to E^{p+r,q+1-r}$ and $d^r : E_{pq} \to E_{p-r,q+r-1}$ respectively satisfying $d^2 = 0$. We call d a differential.*
(b) A spectral sequence is a collection of differential bigraded R-modules $\{E_r^{**}, d_r\}$ and $\{E^r_{**}, d^r\}$ respectively, $r \in \mathbb{N}$, such that $\| d_r \| = (r, 1-r), \| d^r \| = (-r, r-1)$ and

$$E_{r+1}^{**} = H^*(E_r^{**}, d_r),\ E_{**}^{r+1} = H_*(E^r_{**}, d^r).$$

Since we will later need a certain spectral sequence of homology type, we will now confine ourselves to the homological case.

One may think of the E^r-term as a page with lots of dots and arrows. The dots stand for the entry E^r_{pq} and the arrows are of course our differentials. Once a dot is hit by an arrow, it will mapped to zero on the next page. Thus, when working with spectral sequences, we are mainly concerned with when the entries go to zero. Hopefully, we find a page from that on all the pages look the same, that is $E^s = E^{s+1} = \cdots = E^\infty$ for some $s \in \mathbb{Z}$. We say then that the spectral sequence collapses. Unfortunately, it is not enough for a spectral sequence to collapse.

Definition 2.0.11 *Let R be a ring and C an R-modul. A filtration $F.C$ is a ascending/descending sequence of submodules*

$$\ldots F_{-1}C \subset F_0C \subset F_1C \subset \cdots \subset F_pC \subset \cdots \subset C \text{ and}$$

$$\ldots F_1C \subset F_0C \subset F_{-1}C \subset \cdots \subset C \text{ respetively.}$$

Definition 2.0.12 *Let $F.C$ be a filtration of C. Then the associated graded object is*

$$gr_n C = \begin{cases} F_nC/F_{n-1}C & (ascending) \\ F_nC/F_{n+1}C & (descending) \end{cases}$$

Definition 2.0.13 *A spectral sequence $\{E^r_{**}\}$ converges to H if*
(1.) H possesses a filtration and
(2.) $gr_n H \cong \oplus_{p+q=n} E^\infty_{pq}$ for this filtration.

Later, we will mainly be considering free graded algebras and fortunately, there is a general statement concerning convergence of spectral sequences involving such algebras.

Lemma 2.0.14 *If there is a spectral sequence converging to H_* as an algebra and the E^∞-term is a free, graded-commutative, bigraded algebra, then H_* is a free, graded commutative algebra isomorphic to $total(E^\infty_{*,*})$, where $(total(E_{*,*}))^n = \bigoplus_{p+q=n} E_{p,q}$.*

Remark: There is a dual statement for free cocommutative coalgebras.

A detailed proof may be found in [19], p.25. It is not difficult, but a little bit lengthy. Essentially, one defines a filtration on total($E^\infty_{*,*}$) by assigning to each generator and thus to each element a weight. Then, one shows that total($E^\infty_{*,*}$) and H_* are isomorphic as algebras by showing that they have isomorphic filtrations. This is done by double induction: on the algebra degree i and on the filtration degree $i-k$.

As a first example, we will consider the Serre spectral sequence for fibrations.

Theorem 2.0.15 *Let G be an abelian group and $F \to E \to B$ be a fibration with B path-connected. Moreover, let this fibering be orientable in the sense that $\pi_1(B)$ acts trivially on $H_*(F;G)$. Then there is a spectral sequence converging to $H_*(E;G)$ with*

$$E^2_{p,q} \cong H_p(B; H_q(F;G)).$$

In particular, we will be interested in a relative version of this spectral sequence. Let $E' \subset E$ be a subspace such that $(p|E') : E' \to B$ is also a fibration and let $F' = F \cap E'$. Then there is a spectral sequence converging to $H_*(E,E';G)$ with $E^2_{p,q} \cong H_p(B; H_q(F,F'))$.

Remark: There is an analogous cohomological version of this spectral sequence with E^2-term $E_2^{p,q} \cong H^p(B; H^q(F;G))$ converging to $H^*(E;G)$.

2.1 The Eilenberg-Moore spectral sequence

Now, we want to consider the Eilenberg-Moore spectral sequence. As a preparation, we have to discuss the Tor functor.

Let R be a commuative ring and let A, B be R-algebras. In order to calculate $\mathrm{Tor}_n^R(A, B)$, one chooses a resolution $\cdots \to F_1 \to F_0 \to A \to 0$ of A by free (right) R-modules and then tensors this over R with B. Dropping the final term $A \otimes_R B$, one gets a chain complex $\cdots \to F_1 \otimes_R B \to F_0 \otimes_R B \to 0$ whose n^{th} homology group is $\mathrm{Tor}_n^R(A, B)$. Of course, this notation is only justified if $\mathrm{Tor}_n^R(A, B)$ does only depend on A and B and not on the resolution we choose. This is guaranteed by the following

Lemma 2.1.1 *For any two free resolutions F, F' of A there are canonical isomorphisms $H_n(F \otimes B) \cong H_n(F' \otimes B)$.*

Sketch of the proof: The key point is that for free resolutions F, F' of abelian groups H, H' every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F' and that there is only one such chain map up to homotopy. What follows is quite simple. If the maps α_n form the chain map from F to F', then the maps $\alpha_n \otimes id$ form the chain map (again unique up to homotopy) from $F \otimes B$ to $F' \otimes B$. Passing to homology, this chain map induces homomorphisms $\alpha_* : H_n(F \otimes B) \to H_n(F' \otimes B)$. Another important property of chain maps is that for a composition $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ the induced homomorphisms satisfy $(\beta\alpha)_* = \beta_*\alpha_*$. In particular, if α is an isomorphism with inverse β and $H = H''$, then α_* is an isomorphism. In our special case of α being the identity map, we thus get a canonical isomorphism $id_* : H_n(F \otimes B) \cong H_n(F' \otimes B)$. (One may have a look at [9] for the required facts about chain maps.)

□

Remark: If the resolution can be chosen in the category of graded R-modules, tensoring with B stays within this category and there is therefore an induced grading of $\mathrm{Tor}_n^R(A, B)$ as a direct sum of its q^th homogenous subgroups $\mathrm{Tor}_{n,q}^R(A, B)$.

Now, we want to explain a multiplication on $\mathrm{Tor}_*^R(A, B)$ in order to understand it as a graded R-algebra. That is, we search for a map $\mathrm{Tor}_i^R(A, B) \otimes \mathrm{Tor}_j^r(A, B) \to \mathrm{Tor}_{i+j}^R(A, B)$. Let P be a free resolution of A. Then we have as a first step

$$\mathrm{Tor}_i^R(A, B) \otimes \mathrm{Tor}_j^R(A, B) = H_i(P \otimes B) \otimes H_j(P \otimes B) \to H_{i+j}(P \otimes B \otimes P \otimes B)$$

by mapping $c \otimes c'$, where c, c' are cycles in $P \otimes B$, to $c \otimes c' \in P \otimes B \otimes P \otimes B$. Evidently, we have $H_{i+j}(P \otimes B \otimes P \otimes B) \cong H_{i+j}(P \otimes P \otimes B \otimes B)$ and the multiplication on B induces a map

$$H_{i+j}(P \otimes P \otimes B \otimes B) \to H_{i+j}(P \otimes P \otimes B).$$

Thus, the essential point in the second step is a map from $P \otimes P$ to P or another resolution P' of A. In our applications, we will have a map $P \otimes P \to P$ inducing a map on homology and finally a map $\mathrm{Tor}_i^R(A,B) \otimes \mathrm{Tor}_j^r(A,B) \to \mathrm{Tor}_{i+j}^R(A,B)$. The general case follows from the Comparison Theorem (see for example [26], p. 35) which supplies a chain map $P \otimes P \to P'$ that is unique up to chain homotopy equivalence.

Definition 2.1.2 *We call a spectral sequence a first-quadrant spectral sequence if its entries are not trivial only for* $p, q \geq 0$.

Theorem 2.1.3 *Let G be a topological group and suppose that X is a right G-space and Y is a left G-space such that the projection $Y \to Y/G$ is a principal bundle. Then there is a first-quadrant spectral sequence with* $E^2_{pq} = Tor_{p,q}^{H_*(G;k)}(H_*(X;k), H_*(Y;k))$ *converging to* $H_*(X \times_G Y; k)$.

A proof may be found in [10], section 3.1.

We will use this spectral sequence in the context of the universal bundle $G \to EG \to BG$: Let $Y = EG$ and $X = *$, G acting trivially on $*$. By definition, $EG \to EG/G = BG$ is a principal bundle. Moreover, $EG \times_G * = EG \times */\sim \cong EG/G = BG$ with $(x,*) \sim (y,*) \Leftrightarrow y = g(x)$. Evidently, this is an easy special case of the situation described in the theorem.

Proposition 2.1.4 *Let G be a connected topological group. Then there is a spectral sequence of coalgebras with* $E^2 \cong Tor^{H_*(G;k)}(k,k)$ *and converging to* $H_*(BG;k)$ *as a coalgebra.*

Have a look at [19], pp. 267,268 for the proof.

The following theorem goes back to Borel, who actually proved it without using spectral sequences.

Theorem 2.1.5 *If G is a connected topological group with $H_*(G;k) \cong \Lambda(x_1, x_2, \dots)$ as an algebra over k, where $\deg x_i$ is odd for all i, then $H^*(BG;k) \cong k[y_1^*, y_2^*, \dots]$ as algebras with* $\deg y_i^* = \deg x_i + 1$.

Proof: We prove this theorem by use of the above proposition. That is, we first calculate $H_*(BG;k)$. To do so, we have to resolve k over $H_*(G;k) \cong \Lambda(x_1, x_2, \dots)$.
In order to do so, we first have to discuss how to understand k as a $\Lambda(x_1, x_2, \dots)$-module. Since k is concentrated in degree zero and the module structure of a

graded module M over a graded ring R has to fulfill $R_iM_j \subset M_{i+j}$, we have no choice but to demand that $\Lambda^0(x_1, x_2, \dots) \cong k$ acts on k by multiplication and $\Lambda^i(x_1, x_2, \dots)$ acts trivially for $i \geq 1$. Let $\epsilon : \Lambda(x_1, x_2, \dots) \to k$ denote this action.
Let us now put for simplicity $R := \Lambda(x_1, x_2, \dots)$ and consider the bar complex of R and k. It is defined as $B_n(R,k) = R \otimes_k \bar{R} \otimes_k \cdots \otimes_k \bar{R} \otimes_k k$ with n factors $\bar{R}$ which the cokernel of the k-module homomorphism $k \to R$ sending $1 \mapsto 1$. Thus, we have in fact $\bar{R} \cong (R)_{\geq 1}$.
The differential $d : B_n \to B_{n-1}$ is defined as $d = \sum_{i=0}^{n} d_i$ where

$$\begin{aligned}
d_0(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= r_0 r_1 \otimes \bar{r}_2 \otimes \cdots \otimes \bar{r}_n \otimes a, \\
d_i(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= \sum_{i=1}^{n-1} (-1)^i r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a, \\
d_n(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= (-1)^n r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_{n-1} \otimes \epsilon(r_n) a.
\end{aligned}$$

As a first step, we show that $d_i d_j = d_{j-1} d_i$ for $i \leq j-1$ and then $d^2 = 0$ by use of it.
Let $i < j-1$:

$$\begin{aligned}
d_i d_j(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= d_i(r_0 \otimes \cdots \otimes \bar{r}_j \bar{r}_{j+1} \otimes \cdots \otimes a) \\
&= r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes \bar{r}_j \bar{r}_{j+1} \otimes \cdots \otimes a \\
&\text{and} \\
d_{j-1} d_i(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= d_{j-1}(r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a) \\
&= r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes \bar{r}_j \bar{r}_{j+1} \otimes \cdots \otimes a,
\end{aligned}$$

where the last equation holds since $\bar{r}_j$ goes to the $(j-1)^s t$ slot when $\bar{r}_i$ and $\bar{r}_{i+1}$ are drawn together. A similar effect will occur when we now consider the case for $i = j-1$:

$$\begin{aligned}
d_i d_j(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= d_i(r_0 \otimes \cdots \otimes \bar{r}_j \bar{r}_{j+1} \otimes \cdots \otimes a) \\
&= r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_j \bar{r}_{j+1} \otimes \cdots \otimes a \\
&= d_{j-1}(r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a) \\
&= d_{j-1} d_i(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a).
\end{aligned}$$

Finally:

$$\begin{aligned}
d^2 = d(\sum_{j=0}^{n+1} (-1)^j d_j) &= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} d_i d_j \\
&\overset{(*)}{=} \sum_{i \leq j-1 \leq n} (-1)^{i+j} d_{j-1} d_i + \sum_{j \leq i \leq n} (-1)^{i+j} d_i d_j \\
&= -\sum_{i \leq k \leq n} (-1)^{i+k} d_k d_i + \sum_{j \leq i \leq n} (-1)^{i+j} d_i d_j = 0,
\end{aligned}$$

where equation $(*)$ uses $d_i d_j = d_{j-1} d_i$ for $i \leq j-1$. Thus, the bar complex with differential d defined as above is indeed a chain complex.
Now we claim the $B_n(R,k)$ is a resolution of k over R, that is we claim the following sequence is exact:

$$\cdots \xrightarrow{d} R \otimes \bar{R} \otimes \cdots \otimes \bar{R} \otimes k \xrightarrow{d} \cdots \xrightarrow{d} R \otimes \bar{R} \otimes k \xrightarrow{d} R \otimes k \xrightarrow{\epsilon} k \longrightarrow 0\,.$$

We prove this by proving that id_{B_*} is nullhomotopic. That is, we show that $s : B_n \longrightarrow B_{n+1}$,

$$s(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) = \begin{cases} 1 \otimes \bar{r}_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a & \deg r_0 > 0 \\ 0 & \deg r_0 = 0 \end{cases}$$

and $s(a) = 1 \otimes a$ fulfills $ds + sd = id$.
As a first step, let us consider $d \circ s$:

$$\begin{aligned} d \circ s(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= d(1 \otimes \bar{r}_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) \\ &= r_0 \otimes \bar{r}_1 \otimes \ldots \bar{r}_n \otimes a \\ &\quad + \sum_{i=0}^{n-1} (-1)^{i+1} 1 \otimes \bar{r}_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a \\ &\quad + (-1)^{n+1} 1 \otimes \bar{r}_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_{n-1} \otimes \epsilon(r_n) a \end{aligned}$$

Secondly, consider $s \circ d$:

$$\begin{aligned} s \circ d(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) &= s(r_0 r_1 \otimes \bar{r}_2 \otimes \cdots \otimes \bar{r}_n \otimes a) \\ &\quad + s(\sum_{i=1}^{n-1} (-1)^i r_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a) \\ &\quad + s((-1)^n r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_{n-1} \otimes \epsilon(r_n) a) \\ &= 1 \otimes \bar{r}_0 \bar{r}_1 \otimes \bar{r}_2 \otimes \cdots \otimes \bar{r}_n \otimes a \\ &\quad + \sum_{i=1}^{n-1} (-1)^i 1 \otimes \bar{r}_0 \otimes \cdots \otimes \bar{r}_i \bar{r}_{i+1} \otimes \cdots \otimes a \\ &\quad + (-1)^n 1 \otimes \bar{r}_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_{n-1} \otimes \epsilon(r_n) a \end{aligned}$$

Evidently, the last terms in both equations sum up to zero. Moreover, by comparison of the middle terms we see that they sum up to zero as well (except for one summand) due to opposite signs. The only summand left in the first equation is the one for $i = 0$. However, this one and the first term in the second equation also sum up to zero. Thus,

$$(d \circ s + s \circ d)(r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a) = r_0 \otimes \bar{r}_1 \otimes \cdots \otimes \bar{r}_n \otimes a$$

that is $ds + sd = \mathrm{id}$ for $n \geq 1$. At the bottom of the resolution, we have $d \circ s(a) = d(1 \otimes a) = a$ and $s \circ d(a) = s(0) = 0$. Thus, $ds + sd = \mathrm{id}$ as we wanted to show.

We now want to calculate the Tor-term by use of this resolution.
Note that

$$\Lambda(x_1, x_2, \dots) \cong \Lambda(x_1) \otimes_k \Lambda(x_2) \otimes_k \dots$$

as algebras. Moreover, it is $\mathrm{Tor}_*^{\Lambda(x_i)}(k,k) \otimes \mathrm{Tor}_*^{\Lambda(x_j)}(k,k) \cong \mathrm{Tor}_*^{\Lambda(x_i)\otimes\Lambda(x_j)}(k,k)$ as coalgebras (see [19], p. 247). This allows us to calculate the Tor-term via

$$\mathrm{Tor}_*^{\Lambda(x_1,x_2,\dots)}(k,k) \cong \varinjlim \bigotimes_{i=1}^{n} \mathrm{Tor}_*^{\Lambda(x_i)}(k,k).$$

We have

$$\mathrm{Tor}_*^{\Lambda(x_i)}(k,k) = H_*(k \otimes_{\Lambda(x_i)} B_*(\Lambda(x_i), k)).$$

Therefore, consider

$$\begin{aligned} k \otimes_{\Lambda(x_i)} B_n(\Lambda(x_i), k) &= k \otimes_{\Lambda(x_i)} \Lambda(x_i) \otimes_k \bar{\Lambda}(x_i) \otimes_k \cdots \otimes_k \bar{\Lambda}(x_i) \otimes_k k \\ &\cong \bar{\Lambda}(x_i) \otimes_k \cdots \otimes_k \bar{\Lambda}(x_i) \text{ (n times)} \end{aligned}$$

where $\bar{\Lambda}(x_i)$ is the vector space generated by x_i and so $\bar{\Lambda}(x_i) \otimes_k \cdots \otimes_k \bar{\Lambda}(x_i)$ has dimension one over k generated by $x_i \otimes \cdots \otimes x_i$.
Because of our module structure and $x^2 = 0$, the differential $\mathrm{id}\otimes d$ becomes zero and thus

$$\mathrm{Tor}_*^{\Lambda(x_i)}(k,k) \cong \bar{\Lambda}(x_i) \otimes_k \cdots \otimes_k \bar{\Lambda}(x_i).$$

Let us now consider the multiplicative structure on $\mathrm{Tor}^{\Lambda(x_i)}(k,k)$. The bar resolution comes in fact with a product, the so called shuffle product (see for example [26], p.181), where a (p,q)-shuffle of integers $p, q \geq 0$ is a permutation σ of the set $\{1, 2, \dots, p+q\}$ of integers such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.
Before we explain the shuffle product on $B_*(\Lambda(x_i), k)$, note that the latter has a bidegree, i.e. there is a generator y_i of bidegree $(|x_i|, 1)$ or total degree $|x_i| + 1$.
Now define $\gamma_m(y_i) := y_i \otimes \cdots \otimes y_i$ (m times) and the shuffle product $*$ on $B_*(\Lambda(x_i), k)$ as $* : B_p \otimes B_q \to B_{p+q}$,

$$\gamma_p(y_i) * \gamma_q(y_i) = \sum_{(p,q)-\text{shuffles } \sigma} (-1)^{s(\sigma)} \gamma_{p+q}(y_i),$$

where $s(\sigma)$ is a sum over $\deg x_i + 1$ (see [19], p.247) which we do not have to know in detail since all x_i have odd degree and thus, $s(\sigma)$ is always even. Consequently

$$\gamma_p(y_i) * \gamma_q(y_i) = \binom{p+q}{q} \gamma_{p+q}(y_i),$$

where $\binom{p+q}{q}$ is the number of (p,q)-shuffles. Thus, $\mathrm{Tor}_*^{\Lambda(x_i)}(k,k) \cong \Gamma(y_i), |y_i| = |x_i| + 1$, as this is exactly the product structure on a divided power algebra.

Moreover, this is an isomorphism of Hopf algebras, since the comultiplication of the bar resolution is given by

$$\Delta(\gamma_k(y_i)) = \sum_{j=0}^{k} \gamma_j(y_i) \otimes \gamma_{k-j}(y_i)$$

where $\gamma_0(y_i) = 1$.
Finally,

$$\mathrm{Tor}_*^{\Lambda(x_1,x_2,\dots)}(k,k) \cong \varinjlim \bigotimes_{i=1}^{n} \Gamma(y_i) \cong \Gamma(y_1, y_2, \dots)$$

That is, $E^2_{p,q} \cong \Gamma(y_1, y_2, \dots)$ with $|y_i| = |x_i| + 1$ where p corresponds to the external and q to the internal grading.

q						
$\dots$		$\dots$	$\dots$	$\dots$		
$\lvert x_1\rvert + \lvert x_2\rvert + \lvert x_3\rvert$				$x_1x_2x_3$		
$\lvert x_3\rvert$		x_3				
$\lvert x_1\rvert + \lvert x_2\rvert$			x_1x_2			
$\lvert x_2\rvert$		x_2				
$\lvert x_1\rvert$		x_1				
	k					
	$(0,0)$	1	2	3	$\dots$	p

Where there is no entry we mean of course zero.
We see that every element of totalE^2 is of even degree (recall that all x_i are of odd degree!). Since all differentials $d^r : E_{pq} \to E_{p-r,q+r-1}$ decrease total degree by one, all differentials must be zero. That is, the spectral sequence collapses at E^2 and $E^2 \cong E^\infty$. By lemma (2.0.14), the spectral sequence converges to $total(E^\infty_{*,*}) \cong \Gamma(y_1, y_2, \dots)$ with $|y_i| = |x_i| + 1$.

This gives $H_*(BG;k) = \Gamma(y_1, y_2, \dots)$. Since all elements of $H_*(BG;k)$ lie in even dimension, we have

$$H^*(BG;k) \cong H_*(BG;k)^{\mathrm{dual}} \cong \Gamma(y_1, y_2, \dots)^{\mathrm{dual}}$$

as algebras. In order to finish our proof, we have to prove that the dual of the coalgebra $\Gamma(y_1, y_2, \dots)$ is the polynomial algebra $k[y_1^*, y_2^*, \dots]$ with y_i^* dual to y_i and $|y_i^*| = |y_i|$. However, this is due to the fact that they are dual as Hopf algebras. (The finite case is dicussed in the appendix, section (6.2). For infinitely many generators consider the colimits of the algebras on finitely many generators). $\square$

In the later proof, we want to apply our theorems on the Eilenberg-Moore spectral sequence to loop spaces. However, they are only valid for topological groups. Thus, we will now do a little excursion into the simplicial world in order to see how we can understand a loop space as a group.

Chapter 3

The Kan loop group

Definition 3.0.6 *Let $\mathbf{\Delta}$ be the category of finite ordinal numbers with order-preserving maps between them. That is, the objects consist of elements $[n], n \geq 0$, where $[n]$ is a totally ordered set with $n+1$ elements, and the morphisms $\theta : [n] \rightarrow [m]$ satisfy $\theta(i) \geq \theta(j)$ for $i > j$.*

Important examples of morphisms are the so called faces δ_i and degeneracies σ_j: For $0 \leq i, j \leq n$, $\delta_i : [n-1] \rightarrow [n]$ is an injection missing i and $\sigma_j : [n+1] \rightarrow [n]$ is a surjection sending both j and $j+1$ to j.

Definition 3.0.7 *A simplicial object B in a category $\mathcal{C}$ is a contravariant functor $B : \mathbf{\Delta} \rightarrow \mathcal{C}$ or a covariant functor $B : \mathbf{\Delta}^{op} \rightarrow \mathcal{C}$. A covariant functor $\mathbf{\Delta} \rightarrow \mathcal{C}$ is called cosimplicial object in $\mathcal{C}$.*

Our main examples are simplicial objects in the category of sets, i.e. simplicial sets, and simplicial objects in the category of groups, i.e. simplicial groups.

There is another description of simplicial objects, which is equivalent to the above definition (see for example [7], p.4) but more concrete. To understand it, one has to know the following:

Theorem 3.0.8 *For any morphism $\theta : [n] \rightarrow [m]$ there is a unique decomposition*

$$\theta = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_r}\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_s}$$

such that $i_1 \leq i_2 \leq \cdots \leq i_r$ and $j_1 \leq j_2 \leq \cdots \leq j_s$ with $m = n - s + r$. (If the set of indices is empty, then θ is the identity.)

(See [17], p. 453.)

Proposition 3.0.9 *A simplicial object B is a set of objects $B_n, n \geq 0$ in $\mathcal{C}$ together with a set of morphisms $d_i : B_n \rightarrow B_{n-1}, s_j : B_n \rightarrow B_{n+1}, 0 \leq i, j \leq n$*

for all $n \geq 0$*, satisfying the following formulas*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \text{ for } i < j, \\ s_i s_j &= s_{j+1} s_i \text{ for } i \leq j, \\ d_j s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j, \\ id_{B_n} & \text{for } i = j, i = j+1, \\ s_j d_{i-1} & \text{for } i > j+1. \end{cases} \end{aligned}$$

Here, $B_n = B([n]), d_i = \delta_i^*$ *and* $s_j = \sigma_j^*$*. The elements of* B_n *are called* n*-simplices of* B*.*

The most important example of a cosimplicial object is the following:

Example 3.0.10 *Let* $\mathcal{T}$ *denote the category of topological spaces. There is a standard covariant functor* $\mathbf{\Delta} \to \mathcal{T}, [n] \mapsto |\Delta^n|$ *where*

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\} \subset \mathbb{R}^{n+1}$$

is the topological standard n*-simplex with subspace topology.*

Definition 3.0.11 *For any topological space* X*, the singular set* $Sing(X)$ *is the simplicial set given by* $[n] \mapsto \{f : |\Delta^n| \to X|\ f \text{ is continuous}\}$*. In the case of pointed topological spaces we require* $f(*) = f(|\Delta^0|) = x_0$*, where* x_0 *is the basepoint.*

Remark: For a pointed space X, $Sing(X)$ is reduced, that is the set of zero-simplices consists of a single element: $Sing(X)_0 = \{f : |\Delta^0| \to x_0\}$.

$Sing(\ )$ is a functor from the category of topological spaces to the category of simplicial sets. Conversely, there is a functor from simplicial sets to topological spaces which is called geometric realization.

Definition 3.0.12 *Let* A *be a simplicial set. Its geometric realization is defined as the space*

$$|A| = \bigsqcup_{n \geq 0} A_n \times |\Delta^n| / \sim$$

where $\sim$ *is generated by* $(a, \theta_*(t)) = (\theta^*(a), t)$ *for any* $a \in A_n, t \in |\Delta^m|$ *and any* $\theta : [m] \to [n]$ *in* $\mathbf{\Delta}$*. (Recall that* $|\Delta^n|$ *is cosimplicial!).*

Remark: We call an element $a \in A_n$ degenerate if there is some $a' \in A_{n-1}$ such that $a = s_j(a')$ for some j. The geometric realization of A is then, as a CW-complex, a union of cells, which are in bijection with the non-degenerate simplices. The face operators tell us how these cells are glued together. (See [17], p. 455.)

Example 3.0.13 $|Sing(X)| \simeq X$.

Proposition 3.0.14 *There is a bijection*

$$hom_{top}(|A|, X) \cong hom_{simpl}(A, SingX)$$

which is natural in simplicial sets A and topological spaces X. That is, the realization and the singular functor are adjoint.

(See for example [7], p.7.)

Definition 3.0.15 *We define the standard n-simplex in the category of simplicial sets as $\Delta^n = hom_\Delta(\ , [n])$.*

Obviously, our notation gives the impression that the geometric realization of the n-simplex in the category of simplicial sets is the topological standard n-simplex and this is of course true ([7], p. 8).

Δ^n contains subcomplexes $\partial\Delta^n$, called the *boundary*, and Λ^n_k, called the k^th *horn*:

Definition 3.0.16 *$\partial\Delta^n$ is the smallest subcomplex of Δ^n containing all faces $\delta_j(\iota_n), 0 \leq j \leq n$ of the standard simplex $\iota_n = 1_n \in hom_\Delta([n],[n])$. The j-simplices of $\partial\Delta^n$ are*

$$\partial\Delta^n[j] = \begin{cases} \Delta^n[j] & \text{if } 0 \leq j \leq n-1, \\ \text{iterated degeneracies of elements of} & \\ \ \Delta^n[k], 0 \leq k \leq n-1, & \text{if } j \geq n. \end{cases}$$

We write $\partial\Delta^0 = \emptyset$ where $\emptyset$ is the 'unique' simplicial set which consists of the empty set in each degree.
The k^th horn $\Lambda^n_k, n \geq 1$ is the subcomplex of Δ^n which is generated by all faces $d_j(\iota_n)$ except for the k^th face $d_k(\iota_n)$. For example, one could represent Λ^2_1 by the picture

$$\Lambda^2_1 = \begin{tikzcd} & 2 & & & & 2 & & \\ 0 \arrow[rr] & & 1 \arrow[ul] & \subset & 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} = \Delta^2$$

As an easy example of $\partial\Delta^n$, we want now define the simplicial one-sphere as $\mathbb{S}. = \Delta^1/\partial\Delta^1$. To do so, we have to understand $\partial\Delta^1$ in order to see that this quotient is again a simplicial set. From the definition we have

$$\begin{aligned} \partial\Delta^1[0] &= \Delta^1[0] \text{ and} \\ \partial\Delta^1[j] &= \{\text{iterated degeneracies of elements of } \Delta^1[0]\} \text{ for } j \geq 1. \end{aligned}$$

The latter means that we lift all elements of $\Delta^1[0]$ to higher degrees by applying the maps s_i to them. However, if we later consider the geometric realization of $\partial\Delta^1$, these simplices are irrelevant since the geometric realization only sees non-degenerate elements.
Hence, let us have a look at $\Delta^1[0] = hom_\Delta([0],[1])$. Obviously, we have only two maps

$$\begin{tikzcd} & 1 & & 1 \\ 0 \arrow[r] & 0 & 0 \arrow[ur] & 0 \end{tikzcd}$$

and none of them is a degeneracy.
If we now define the quotient $\Delta^1/\partial\Delta^1$ levelwise, that is $(\Delta^1/\partial\Delta^1)[j] = \Delta^1[j]/\partial\Delta^1[j]$, we get again a simplicial set.

However, to see that our definition $\mathbb{S}^1. = \Delta^1/\partial\Delta^1$ makes sense we have to check that $|\mathbb{S}.| = \mathbb{S}^1$. A first step to see this is knowing, that a left adjoint functor preserves colimits ([26], p.55). We already saw in (3.0.14), that $|\ |$ is left adjoint and a quotient can be considered as colimit of the inclusion. Thus, we need to know $|\Delta^1|$ and $|\partial\Delta^1|$. In particular, we have to check that the geometric realization of the simplicial boundary is the topological boundary!
Let us consider Δ^1. We already explained how the 0-simplices look like. The 1-simplices ($hom_\Delta([1],[1])$) are the following:

$$\begin{tikzcd} 1 \arrow[r] & 1 & 1 \arrow[r] & 1 & 1 \arrow[dr] & 1 \\ 0 \arrow[r] & 0 & 0 \arrow[ur] & 0 & 0 \arrow[r] & 0 \end{tikzcd}$$

The last one is a degeneracy and the middle one is degenerate as image of a 0-simplex. All higher simplices $\Delta^1[n], n \geq 2$ are represented by non-injective maps and thus degenerate. Hence, by definition of the geometric realization and the following remark, we have $|\Delta^1| =$ —— .
We already explained that $\partial\Delta^1$ only consists of 0-simplices. More precisely, it consists of two 0-simplices i.e. two simplicial points. By definition of the realization, we trivially get $|\partial\Delta^1| =$ · · . Hence, $|\mathbb{S}^1.| = |\Delta^1|/|\partial\Delta^1| =$ ◯ $= \mathbb{S}^1$.

Definition 3.0.17 *A Kan complex is a simplicial set A such that the canonical map $A \to *$ is a Kan fibration. That is, for every k and for every commutative diagram of simplicial set homomorphisms*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "i_k"] & A \arrow[d] \\ \Delta^n \arrow[r] & * \end{tikzcd}$$

there is a map $\theta : \Delta^n \to A$ *making the diagram commute. Here,* i_k *is of course the inclusion* $\Lambda^n_k \subset \Delta^n$*.*

Lemma 3.0.18 *For each space* X*, the map* $Sing(X) \to *$ *is a Kan fibration and thus* $Sing(X)$ *is always a Kan complex.*

(See [7], p. 11.)

Definition 3.0.19 *Let* A, B *be simplicial sets. We then define* $Map\ (A, B)_n = hom_{simpl}(A \times \Delta^n, Y)$ *and take face and degeneracy maps to be induced by the standard maps between the* Δ^n*. This makes* $Map\ (A, B)$ *again a simplicial set.*

With this we define the simplicial loop space $\Omega^{simpl} A = Map\ (\mathbb{S}., A)$.

For topological spaces, we define $Map\ (X, Y)$ to be the simplicial set with n-simplices the continous functions $X \times |\Delta^n| \to Y$ and face and degeneracy maps induced by the standard maps between the Δ^n.

Lemma 3.0.20 *With the above definitions, proposition* (3.0.14) *extends to a natural isomorphism of simplicial mapping spaces*

$$Map\ (|A|, X) \cong Map\ (A, Sing(X))$$

for a simplicial set A *and a topological space* X*. Moreover, the geometric realization of a simplicial mapping space is natural isomorphic to a topological mapping space.*

(See [11], pp. 7,8.)

Theorem 3.0.21 *Let* A *be a reduced Kan complex. Then there is a simplicial group* GA *such that* $GA \simeq \Omega^{simpl} A$ *is a weak equivalence (in the sense that we get an isomorphism on homotopy) which is natural in* A*.*

GA is called the *loop group of* A for obvious reasons. Since the construction goes back to Kan, it is often called Kan loop group. Its construction is can be found in [7], section V.5, together with the proof of theorem. The main point of this theorem is that the loop space of a reduced Kan complex can be regarded as a group. Moreover, the geometric realization of a simplicial group is in fact a topological group: Because of $|GA| \times |GA| \cong |GA \times GA|$ ([7], p. 9), we get a continuous map ($|\ |$ is a functor!) $|GA| \times |GA| \to |GA|$.

As we said above, $Sing(X)$ is a reduced Kan complex for any topological space X. Thus, $\Omega^{Simpl} Sing(X)$ is weakly equivalent to a simplicial group (depending on X). Lemma (3.0.20) guarantees

$$\begin{aligned} |\Omega^{simpl} Sing(X)| &= |Map\ (\mathbb{S}^1., Sing(X))| \\ &\overset{lm.(3.0.20)}{\cong} |Map\ (|\ \mathbb{S}^1.|, X)| \\ &\cong \hom_{top}(\mathbb{S}^1, X) = \Omega X. \end{aligned}$$

On homology this gives

$$H_*(\Omega X) \cong H_*(|\Omega^{simpl} Sing(X)|) \overset{thm(3.0.21)}{\cong} H_*(|GSing(X)|).$$

Hence, the homology of ΩX is isomorphic to the homology of a topological group. If we later apply our theorems on the Eilenberg-Moore spectral sequence to ΩX, we will actually replace ΩX by the topological group $|GSing(X)|$, calculate their homology and then take advantage of the above isomorphism.

Chapter 4

Cohomology Operations

In general, cohomology operations are a tool that gives information about a space X. Let us say we want to show that two spaces are of a different homotopy type. One possibility would be to compare their cohomology groups and show that they differ in some degree. Here, primary cohomology operations come into play: They are maps of a certain degree on the singular cohomology groups of a space X. However, it may happen that a composition of such maps is zero and thus fail to give further information about the underlying space. In this case, one can construct new operations - secondary cohomology operations - which rescue some of the information the primary operations lost.

4.1 Primary Operations

Definition 4.1.1 *Let $H^n(\ ;\pi)$ and $H^q(\ ;G)$ be the singular cohomology functors from the category of topological pairs and continous maps to the category of sets and fundtions, with n and q positive.*
For $n, q > 0$, a primary cohomology operation θ of type (π, n, G, q) is a natural transformation from $H^n(\ ;\pi)$ to $H^q(\ ;G)$.

Thus, for any pair (X, Y) we have a function

$$\theta(X, Y) : H^n(X, Y; \pi) \longrightarrow H^q(X, Y; G)$$

and for any map of pairs $f : (X, Y) \to (W, Z)$, we have

$$\theta(X, Y) \circ f^* = f^* \circ \theta(W, Z)$$

where f^* is the map induced on cohomology.

Consequences of naturalitiy
• $\theta(X, Y)$ is a pointed map of pointed sets.
As each $H^n(X, Y; \pi)$ is an abelian group, it has a distinguished zero element,

namely $H^n(*\ ;\pi)$ which we understand as an element in $H^n(X,Y;\pi)$ via $p_* : H^n(*;\pi) \to H^n(X,Y;\pi)$ induced from $p : (X,Y) \to *$. It follows from naturality that θ is as asserted.
• $\theta(X,X)$ is the zero map for $q < n$.
Starting with a CW pair, we observe that the q-th cohomology of the pair maps monomorphically into the q-th cohomology of the q-skeleton of the pair. As the n-th cohomology of the q-skeleton is zero, θ has to be the zero map by naturality. For arbitrary pairs, this follows by use of the CW approximation theorem.

Definition 4.1.2 *Let $\theta_n : n \geq 1$ be a sequence of cohomology operations of type $(\pi, n, G, n+i)$ for a fixed positive integer i. We call such a sequence a stable cohomology operation of degree i provided the following diagram commutes for each pair (X,Y) and each $n \geq 1$:*

$$\begin{tikzcd} H^n(Y;\pi) \arrow[r, "\delta"] \arrow[d, "\theta_n(Y)"'] & H^{n+1}(X,Y;G) \arrow[d, "\theta_{n+1}(X,Y)"] \\ H^{n+i}(Y;\pi) \arrow[r, "\delta"] & H^{n+i+1}(X,Y;G) \end{tikzcd}$$

We call the individual θ_n the components of a stable operation.

4.1.1 Steenrod Operations

In the following, we will discuss an important example of primary cohomology operations: Steenrod operations. They are definded on cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients and are named Steenrod squares for $p = 2$ and reduced powers for p odd. While the reduced powers will turn up in a theorem we will need later, the Steenrod squares will only serve to explain the secondary operations.
Unfortunately, the contstruction of these operations is quite lengthy. Hence, we will only present their properties. A nice reference for the construction is [9], section 4.L.

Proposition 4.1.3 *There is a stable operation Sq^i having components of type $(\mathbb{Z}/2\mathbb{Z}, n, \mathbb{Z}/2\mathbb{Z}, n+i)$. That is, we have maps*

$$Sq^i : H^n(X,Y;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{n+i}(X,Y;\mathbb{Z}/2\mathbb{Z})$$

which commute with the connecting homomorphism.

Properties
(1) $Sq^0 =$id
(2) If $n < i$, then Sq^i is the zero map.

(3) If $n = i$, then Sq^n is the cup product $Sq^n x = x^2$.
(4) Cartan formula: On cup products, Sq^i satisfies the equation

$$Sq^i(x \cup y) = \sum_{i=j+k} Sq^j x \cup Sq^k y$$

(5) Adem relations: If $0 < a < 2b$, then

$$Sq^a Sq^b = \sum \binom{b-1-t}{a-2t} Sq^{a+b-t} Sq^t$$

with non-zero summands only for $0 \leq t \leq a/2$.

Lemma 4.1.4 *For each i, Sq^i commutes with (unreduced) suspension. That is, the following diagram commutes:*

$$\begin{array}{ccc}
\tilde{H}^n(X;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\Sigma} & \tilde{H}^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z}) \\
\downarrow{\scriptstyle Sq^i} & & \downarrow{\scriptstyle Sq^i} \\
\tilde{H}^{n+i}(X;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\Sigma} & H^{n+i+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})
\end{array}$$

Proposition 4.1.5 *There are stable operations, called reduced powers of type $(\mathbb{Z}/p\mathbb{Z}, n, \mathbb{Z}/p\mathbb{Z}, n+2i(p-1))$ for p an odd prime:*

$$\mathcal{P}^i : H^n(X,Y;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{n+2i(p-1)}(X,Y;\mathbb{Z}/p\mathbb{Z}).$$

In addition, there is the Bockstein homomorphism

$$\beta : H^n(X,Y;\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^{n+1}(X,Y;\mathbb{Z}/p\mathbb{Z})$$

obtained from the coefficient sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{*p} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \to 0.$$

The Bockstein operation is not stable. However, the signed Bockstein $B = (-1)^n\beta$ for β defined on $H^n(X,Y;\mathbb{Z}/p)$ is a stable operation.

Properties
(1) $\mathcal{P}^0$ =id
(2) If $n = 2i$, then $\mathcal{P}^i x = x^p$ for any cohomology class x of dimension n.
(3) If $n < 2i$, then $\mathcal{P}^i x = 0$ for any cohomology class x of dimension n.
(4) Cartan Formula: $\mathcal{P}^i(x \cup y) = \sum_{j+k=i} \mathcal{P}^j x \cup \mathcal{P}^k y$ and, for the unsigned Bockstein, $\beta(x \cdot y) = \beta x \cdot y + (-1)^{|x|} x \cdot \beta y$. This gives in particular $\beta x^p = 0$.
(5) Adem relations:
If $a < pb$, then

$$\mathcal{P}^a \mathcal{P}^b = \sum (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} \mathcal{P}^{a+b-t} \mathcal{P}^t$$

with non-zero summands only for integers t satisfying $0 \leq t \leq a/p$.
If $a \leq pb$, then

$$\begin{aligned} \mathcal{P}^a \beta \mathcal{P}^b &= \sum (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta \mathcal{P}^{a+b-t} \mathcal{P}^t \\ &+ \sum (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} \mathcal{P}^{a+b-t} \beta \mathcal{P}^t. \end{aligned}$$

Moreover, $\beta^2 = 0$.

The Steenrod Algebra

The Steenrod operations modulo Adem relations form an algebra, the Steenrod algebra $\mathcal{A}_p$. It has the nice property that for every space X and every prime p, $H^*(X; \mathbb{Z}/p)$ is a module over $\mathcal{A}_p$.

Definition 4.1.6 *The Steenrod algebra $\mathcal{A}_2$ is defined to be the algebra over $\mathbb{Z}/2$ that is the quotient of the algebra of polynomials in the noncommuting variables $Sq^1, Sq^2, \ldots$ by the twosided ideal generated by the Adem relations. Similarly, the Steenrod algebra $\mathcal{A}_p$ for odd primes is defined to be the algebra over $\mathbb{Z}/p$ formed by polynomials in the noncommuting variables $\beta, P^1, P^2, \ldots$ modulo the Adem relations.*
The Steenrod algebra is a graded algebra , the elements of degree k being those that map $H^n(X; \mathbb{Z}/p)$ to $H^{n+k}(X; \mathbb{Z}/p)$ for all n.

As the next proposition shows us, $\mathcal{A}_2$ is generated as an algebra by the elements Sq^{2^k}, while $\mathcal{A}_p$ for p odd is generated by β and elements $\mathcal{P}^{p^k}$.

Proposition 4.1.7 *If i is not a power of 2, there is a relation $Sq^i = \sum_{0<j<i} a_j Sq^{i-j} Sq^j$ with coefficients $a_j \in \mathbb{Z}/2$. Similarly, if i is not a power of p, there is a relation $P^i = \sum_{0<j<i} a_j P^{i-j} P^j$ with $a \in \mathbb{Z}/p$. These operations are called decomposable.*

Example 4.1.8 $Sq^5 = Sq^1 Sq^4$ *and* $Sq^6 = Sq^2 Sq^4 + Sq^5 Sq^1$.

4.2 Secondary Operations

The presentation here mainly follows the one in [8].

Let $C_0 \xrightarrow{\theta} C_1 \xrightarrow{\varphi} C_2$ be a pair of composable maps with C_2 being simply connected.
Given a space X, let $S_\theta(X)$ denote the set of homotopy classes of maps $\epsilon : X \to C_0$ such that the composition $\theta \circ \epsilon$ is null-homotopic,

$$S_\theta(X) = \{[\epsilon] |\ \epsilon : X \to C_0,\ \theta \circ \epsilon \sim *\}.$$

Moreover, let $T_{\Omega\varphi}(X)$ denote the quotient

$$T_{\Omega\varphi}(X) = [X, \Omega C_2]/im\Omega\varphi_{\#},$$

where $\Omega\varphi_{\#} : [X, \Omega C_1] \to [X, \Omega C_2]$ is given by $\Omega\varphi_{\#}(g) = \Omega\varphi \circ g$. We need here the simply connectivity of C_2: If C_2 is simply connected, then ΩC_2 is connected and $T_{\Omega\varphi}(X)$ is well defined.
For illustration, one may consider the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\epsilon"] & C_0 \arrow[r, "\theta"] & C_1 \arrow[rr, "\varphi"] & & C_2 \\
& & \Omega C_1 \arrow[u] \arrow[rr, "\Omega\varphi"] & & \Omega C_2 \arrow[u] \\
& & & X \arrow[ul, "g"] \arrow[ur, "\Omega\varphi_{\#}(g)"'] &
\end{tikzcd}$$

We write $[\![g]\!]$ to denote the image of $g : X \to \Omega C_2$ in $T_{\Omega\varphi}(X)$.
For $f : Y \to X$ we have

$$f^{\#} : S_\theta(X) \to S_\theta(Y) \text{ given by } f^{\#}([\epsilon]) = [\epsilon \circ f]$$

and

$$f^{\#} : T_{\Omega\varphi}(X) \to T_{\Omega\varphi}(Y) \text{ given by } f^{\#}([\![g]\!]]) = [\![g \circ f]\!].$$

Definition 4.2.1 *A secondary cohomology operation Θ is a natural transformation of the functors S_θ and $T_{\Omega\varphi}$. That is, for each $f : X \to Y$, the following diagram commutes:*

$$\begin{tikzcd}
S_\theta(X) \arrow[r, "f^{\#}"] \arrow[d, "\Theta"'] & S_\theta(Y) \arrow[d, "\Theta"] \\
T_{\Omega\varphi}(X) \arrow[r, "f^{\#}"] & T_{\Omega\varphi}(Y)
\end{tikzcd}$$

For a space X, $S_\theta(X)$ is a set and $T_{\Omega\varphi}(X)$ is a group. If X is a point, Θ is the zero map for $T_{\Omega\varphi}(*) = 0$. Since we work with homotopy classes of maps, we have $\Theta(\epsilon) = 0$ for any $\epsilon : X \to C_0$ which is null-homotopic. Thus, Θ is automatically a map of pointed sets.
In the case where both θ and $\Omega\varphi$ are zero maps, our definition will agree with a primary cohomology operation if C_0 and C_2 are Eilenberg-MacLane spaces.

As the submodule used to form the quotient $T_{\Omega\varphi}(X)$ is of some importance and will be referred to later on, we give it a name.

Definition 4.2.2 *We call* $\mathrm{im}\Omega\varphi_{\#}$, $\Omega\varphi_{\#} : [X, \Omega C_1] \to [X, \Omega C_2]$, *the indeterminacy of Θ and write it as* $\mathrm{Ind}(\Theta, X) = \mathrm{im}\Omega\varphi_{\#}$. *Note that for $f : Y \to X$, we have* $f^{\#}\mathrm{Ind}(\Theta, X) \subset \mathrm{Ind}(\Theta, Y)$.

As we said above, secondary operations become especially important when primary operations fail. As the Adem relations give rise to a lot of null-compositions of primary operations, this will often be the case. Thus, we will explain in the following how to construct secondary compositions for nullhomotopic compositions.

4.2.1 Operations associated to nullhomotopic compositions

Definition 4.2.3 *We write the adjoint of $f : I \times X \to Y$ as $f^\natural : X \to Y^I, f^\natural(x)(t) = f(t,x)$.*

Example 4.2.4 *It is $PX = \{w : I \to X |\ w(0) = *\}$. Given a contracting homotopy $H : I \times X \to Y$, we have $H^\natural : X \to PY,\ H^\natural(x)(s) = H(s,x)$.*

Definition 4.2.5 *The direction reversal map τ of the unit interval is given by $\tau(t) = 1 - t$. When τ is used to reverse direction in homotopies, we write $H_\tau(t,x) = H(\tau(t),x)$.*

Definition 4.2.6 *Given a map $f : B \longrightarrow B_0$ we construct the fiber square*

$$\begin{tikzcd}
W_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PB_0 \arrow[d, "e"] \\
B \arrow[r, "f"] & B_0
\end{tikzcd}$$

with p_i being the projection on the ith component and e the evaluation on 1. W_f is called homotopy fiber of f and its elements are pairs $(b,w) \in B \times PB_0$ such that $f(b) = w(1)$.

Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be a pair of composable maps and H a contracting homotopy from $*$ to $\beta\alpha$. We will call this a sequence with homotopy and denote it by (β, α, H). With this data we get two new maps.

Definition 4.2.7 (Lifting of α) *We obtain a map $\bar{\alpha} : A \to W_\beta$ from pull-back data of the following diagram:*

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "\alpha"'] & A \arrow[r, equal] \arrow[d, "\beta\alpha"] & A \arrow[d, "H^\natural"] \\
B \arrow[r, "\beta"] & C & P_0C \arrow[l, "e_0"'] \\
& W_\beta \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

Assertion: $\bar{\alpha}(a) = (\alpha(a), H(\ ,a))$. To prove that $(\alpha(a), H(\ ,a))$ is in W_β, we have to check that $\beta(\alpha(a)) = H(1,a)$. Since H is a contracting homotopy from $*$ to $\beta\alpha$, this is quite obvious.

Definition 4.2.8 (Colifting of β) *The map $\tilde{\beta} : W_\alpha \to \Omega C$ is obtained from pull-back data of the following diagram*

$$\begin{tikzcd}
 & W_\alpha \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A \arrow[r, "\alpha"] \arrow[d, "H_\tau^\natural"] & B \arrow[d, "\beta"] & P_0B \arrow[l, "e_0"'] \arrow[d, "P_0\beta"] \\
P_1C \arrow[r, "e_1"] & C & P_0C \arrow[l, "e_0"'] \\
 & \Omega C \arrow[ul, "i_1"] \arrow[ur, "i_0"'] &
\end{tikzcd}$$

Assertion: $\tilde{\beta}(a,w)(s) = \begin{cases} \beta w(2s) & 0 \le s \le 1/2, \\ H(2-2s,a) & 1/2 \le s \le 1. \end{cases}$
Recall that $\Omega C = \{(w_1, w_0) \in P_1C \times P_0C |\ e_1(w_1) = w(1) = w(0) = e_0(w_o)\}$. Thus, to prove that $\tilde{\beta}(a,w) \in \Omega C$, we have to check that $\tilde{\beta}(a,w)(1) = \tilde{\beta}(a,w)(0)$. This is quite easy: $\tilde{\beta}(a,w)(1) = H(0,a) = * = \beta w(0) = \tilde{\beta}(a,w)(0)$. Moreover, it is $\tilde{\beta}(a,w)(1/2) = \beta w(1) = \beta(\alpha(a)) = H(1,a)$.

Definition 4.2.9 (Secondary compositions) *Given three composable maps $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$ and contracting homotopies H from $*$ to $\beta\alpha$, K from $*$ to $\gamma\beta$, we can construct a secondary composition*

$$\tilde{\gamma} \circ \bar{\alpha} : A \longrightarrow W_\beta \longrightarrow \Omega D$$

through the homotopy fiber.
From the formulas above we get by composition:

$$(\tilde{\gamma} \circ \bar{\alpha})(a)(s) = \{\gamma H, K_\tau \alpha\} = \begin{cases} \gamma H(2s,a) & 0 \le s \le 1/2, \\ K(2-2s, \alpha(a)) & 1/2 \le s \le 1. \end{cases}$$

Obviously, $(\tilde{\gamma} \circ \bar{\alpha})(a)(1/2) = \gamma H(1,a) = (\gamma \circ \beta \circ \alpha)(a) = K(a, \alpha(a))$.

Definition 4.2.10 *Let H_1, H_2 be contracting homotopies for the composition $\beta\alpha$. We measure their difference by $\delta(H_1, H_2) : A \longrightarrow \Omega C$ given by*

$$\delta(H_1, H_2) = \{H_1, H_{2\tau}\}^\natural = \begin{cases} H_1(2s,a) & 0 \le s \le 1/2, \\ H_2(2-2s,a) & 1/2 \le s \le 1. \end{cases}$$

Definition 4.2.11 *Two sequences with homotopy, (β, α, H) and (β', α', H') are homotopic provided there are homotopies L from α to α' and M from β to β', such that*

$$\delta(H', \{H, \beta L, M\alpha'\}) : A \longrightarrow \Omega C$$

is nullhomotopic.

Proposition 4.2.12 *Given pairs of maps $A \xrightarrow{\alpha,\alpha'} B \xrightarrow{\beta,\beta'} C \xrightarrow{\gamma,\gamma'} D$ and contracting homotopies to form four sequences with homotopy $(\beta, \alpha, H), (\gamma, \beta, K)$, $(\beta', \alpha', H'), (\gamma', \beta', K')$ and given homotopies L from α to α', M from β to β' and N from γ to γ' such that*

$$\delta_1 = \delta(H', \{H, M\alpha, \beta' L\}) : A \to \Omega C$$

and

$$\delta_2 = \delta(K', \{K, N\beta, \gamma' M\}) : B \to \Omega D$$

are nullhomotopic, then the secondary compositions are homotopic: $\tilde{\gamma} \circ \bar{\alpha} \simeq \tilde{\gamma}' \circ \bar{\alpha}'$.

Proof: Recall the explicit formula for $\tilde{\gamma} \circ \bar{\alpha}$ in definition 1.2.6. Together with definition 1.2.7 we obtain

$$\tilde{\gamma} \circ \bar{\alpha} = \delta(\gamma H, K\alpha) = \{\gamma H, K_\tau \alpha\}^\natural = \begin{cases} \gamma H(2s, a) & 0 \leq s \leq 1/2 \\ K\alpha(2-2s, a) & 1/2 \leq s \leq 1 \end{cases}$$

and

$$\tilde{\gamma}' \circ \bar{\alpha}' = \delta(\gamma' H', K'\alpha') = \{\gamma' H', K'_\tau \alpha'\}^\natural = \begin{cases} \gamma' H'(2s, a) & 0 \leq s \leq 1/2 \\ K'\alpha'(2-2s, a) & 1/2 \leq s \leq 1 \end{cases}.$$

respectively.

Saying that δ_1 and δ_2 are nullhomotopic means that the sequences (β, α, H) and (β', α', H') and (γ, β, K) and (γ', β', K') respectively are homotopic.

Thus, if we think of a homotopy H between $f, g : X \to Y$ as a square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] \arrow[r, phantom, shift right=6ex, "\xrightarrow{H}"] & X \arrow[d, "g"] \\ Y \arrow[r, equal] & Y \end{tikzcd} ,$$

then the required homotopies can be deduced from the following schematic diagram:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "K\alpha"'] & A \arrow[rr, equal] \arrow[d, "K\alpha'"] & & A \arrow[d, "K'\alpha'"] \\
D \arrow[r, equal, "\gamma\beta L"] & D \arrow[r, equal, "\gamma M\alpha'"] & D \arrow[r, equal, "N\beta'\alpha'"] & D \\
A \arrow[u, "\gamma H"] \arrow[rr, equal] & & A \arrow[u, "\gamma H'"'] \arrow[r, equal] & A \arrow[u, "\gamma' H'"']
\end{tikzcd}
$$

□

We will now come back to secondary cohomology operations and explain the promised example based on secondary compositions.

Assume there is a contracting homotopoy H for the composable pair $C_0 \xrightarrow{\theta} C_1 \xrightarrow{\varphi} C_2$. Then with $\epsilon : X \to C_0$ representing an element in $S_\theta(X)$, we have a set of secondary compositions $\{\tilde{\varphi} \circ \bar{\epsilon} \mid \bar{\epsilon} : X \to W_\theta \text{ is a lift of } \epsilon\}$. Applying proposition 1.2.1 to $H' = \{H, M\alpha, \beta' L\}$ and $K' = \{K, N\beta, \gamma' M\}$, it follows that these secondary compositions are invariants of the homotopy class of the sequence with homotopy (φ, θ, H). That is, we get a well-defined natural transformation $\Theta : S_\theta(\ ) \to T_{\Omega\varphi}(\ )$ given by

$$\Theta([\epsilon]) = [[\tilde{\varphi} \circ \bar{\epsilon}]]$$

for each homotopy class of (φ, θ, H).

Example 4.2.13 *Consider the following spaces and maps for an arbitrary, but fixed integer $n \geq 1$:*

$$
\begin{aligned}
C_0 &= K(\mathbb{Z}/2\mathbb{Z}, n), \\
C_1 &= K(\mathbb{Z}/2\mathbb{Z}, n+1) \times K(\mathbb{Z}/2\mathbb{Z}, n+2), \\
C_2 &= K(\mathbb{Z}/2\mathbb{Z}, n+4), \\
\theta : C_0 &\longrightarrow C_1 \text{ representing } \begin{pmatrix} Sq^1 \\ Sq^2 \end{pmatrix}, \\
\varphi : C_1 &\longrightarrow C_2 \text{ representing } (Sq^3, Sq^2).
\end{aligned}
$$

From the Adem relations, we know that $Sq^2Sq^2 = Sq^3Sq^1$, thus $Sq^3Sq^1 + Sq^2Sq^2 = 0$ over $\mathbb{Z}/2\mathbb{Z}$. This gives $[\varphi][\theta] = 0$. Because of the representation theorem (1.3.10)*, we then have for a space X that*

$$S_\theta = \{[\epsilon] \mid \epsilon : X \to C_0, \epsilon \circ \theta = 0\} = \{x \in H^n(X; \mathbb{Z}/2\mathbb{Z}) \mid Sq^1(x) = 0 = Sq^2(x)\}.$$

Since $\Omega K(G, n+1) = K(G, n)$*, we have*

$$T_{\Omega\varphi}(X) = H^{n+3}(X; \mathbb{Z}/2\mathbb{Z})/Sq^3H^n(X; \mathbb{Z}/2\mathbb{Z}) + Sq^2H^{n+1}(X; \mathbb{Z}/2\mathbb{Z}),$$

which becomes quite obvious by regarding the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\epsilon"] & K(\mathbb{Z}/2\mathbb{Z}, n) \arrow[r, "\theta"] & K(\mathbb{Z}/2\mathbb{Z}, n+1) \times K(\mathbb{Z}/2\mathbb{Z}, n+2) \arrow[r, "\varphi"] & K(\mathbb{Z}/2\mathbb{Z}, n+4) \\
& & K(\mathbb{Z}/2\mathbb{Z}, n) \times K(\mathbb{Z}/2\mathbb{Z}, n+1) \arrow[u] \arrow[r, "\Omega\varphi"] & K(\mathbb{Z}/2\mathbb{Z}, n+3) \arrow[u] \\
& & & X \arrow[ul] \arrow[u]
\end{tikzcd}$$

4.3 Factorization of primary operations by secondary operations

One main point in the later proof will be the factorization of a certain Steenrod operation by secondary operations. The idea of this factorization goes back to Adams' factorization of the Steenrod square Sq^i for $p = 2$ in [1]. A few years later, Liulevicius proved the factorization in question of the cyclic reduced power $\mathcal{P}^{p^n}$, for $n \geq 0$ and p odd in his Ph. D. thesis (see [16]). We will therefore not be able to give a proof of this factorization, but only state the theorem and explain the operations involved.

Theorem 4.3.1 *Let p be an odd prime. There exist stable secondary cohomology operations $\Psi_i, \mathcal{R}, \Gamma_\gamma$, elements $a_{k,i}, b_k, c_{k,\gamma}$ of positive grading in the Steenrod Algebra $\mathcal{A}$ over the field $\mathbb{Z}/p\mathbb{Z}$ and a scalar $0 \neq \nu_k \in \mathbb{Z}/p\mathbb{Z}$ such that*

$$\sum_{i=1}^{k} a_{k,i}\Psi_i + b_k\mathcal{R} + \sum_{\gamma} c_{k,\gamma}\Gamma_\gamma = \{v_k\mathcal{P}^{p^{k+1}}\}$$

for all integers $k \geq 0$, modulo total indeterminacy.

These operations are as follows:
$\mathcal{R}$ is defined on $S_\theta(X) = \{x \in H^m(x; \mathbb{Z}/p\mathbb{Z}) \mid \beta(x) = 0 = \mathcal{P}^1(x)\}$ and $\mathcal{R}(x)$ is an element of

$$T_{\Omega\varphi}(X) = H^{m+4(p-1)}(X; \mathbb{Z}/p\mathbb{Z}) \Big/ \mathcal{P}^2H^m(X; \mathbb{Z}/p\mathbb{Z}) + (1/2\beta\mathcal{P}^1 - \mathcal{P}^1\beta)H^{m+2p-3}(X; \mathbb{Z}/p\mathbb{Z})$$

with θ representing $\binom{\beta}{\mathcal{P}^1}$ and φ representing $(\mathcal{P}^2, 1/2\beta\mathcal{P}^1 - \mathcal{P}^1\beta)$.
For $k > 0, \Psi_k$ is defined on $S_{\theta'} = \{x \in H^m(X; \mathbb{Z}/p\mathbb{Z}) \mid \beta(x) = 0 = \mathcal{P}^{p^i}(x), i = 0, 1, \ldots, k\}$ and $\Psi_k(x)$ lies in

$$T_{\Omega\varphi'}(X) = H^{m+2p^k(p-1)}(X; \mathbb{Z}/p\mathbb{Z}) \Big/ \mathcal{P}^{p^k}H^m(X; \mathbb{Z}/p\mathbb{Z}) + \sum_{i=0}^{k} \vartheta_i H^{m+2(p^k-p^i)(p-1)-1}(X; \mathbb{Z}/p\mathbb{Z}),$$

where ϑ_i are elements of the Steenrod algebra $\mathcal{A}$ and θ' and φ' represent

$$\begin{pmatrix} \beta \\ (\mathcal{P}^{p^k} - \mathcal{P}^{p^0}, \ldots, \mathcal{P}^{p^k} - \mathcal{P}^{p^k}) \end{pmatrix} \text{ and } (\mathcal{P}^{p^k}, (\mathcal{P}^{p^0}, \ldots, \mathcal{P}^{p^k})).$$

The only thing we need to know about the operations Γ_γ is that they are of odd degree.

Chapter 5

The Brown-Peterson spectrum

5.1 Thom spectra

We now construct a so called Thom spectrum $M(f)$ for each map $f : X \to B\mathcal{F}$ where the latter is defined as the telescope of the sequence $\{\cdots \to B\mathcal{F}_n \to B\mathcal{F}_{n+1} \to \dots\}$ and $B\mathcal{F}_n$ is the classifying space for $(\mathbb{S}^n, *)$-fibrations.

Our presentation relates to the one of Rudyak in [24].

Definition 5.1.1 *Given a diagram $\eta_1 \xleftarrow{\phi_1} \xi \xrightarrow{\phi_2}$ of morphisms over $B = base(\xi) = base(\eta_i), i = 1, 2,$ we define its double mapping cylinder over B to be the bundle*

$$DCyl(\phi_1, \phi_2) := \xi \times [0, 2] \cup_\psi (\eta_1 \sqcup \eta_2),$$

where $\psi : (\xi \times \{0\}) \sqcup (\xi \times \{2\}) = \xi \sqcup \xi \xrightarrow{\phi_1 \sqcup \phi_2} \eta_1 \sqcup \eta_2$.

With this we define the homotopy smash product $\xi \wedge^h \eta$ of two sectioned bundles $(\xi, s_\xi), (\eta, s_\eta)$ as a certain double mapping cylinder. Its construction can be found in [24], pp. 188,189. It is rather complicated and provides not much insight, so we omit it here.

Definition 5.1.2 *Let $B\mathcal{F}_n$ be the classifying space for $(\mathbb{S}^n, *)$-fibrations and $\gamma^n_\mathcal{F}$ the universal object over it. Let θ be the trivial $(\mathbb{S}^1, *)$-fibration over a point and $\rho^n_\mathcal{F} : \gamma^n_\mathcal{F} \wedge^h \theta \to \gamma^{n+1}_\mathcal{F}$ the classifying morphism for $\gamma^n_\mathcal{F} \wedge^h \theta$. We set $r^n_\mathcal{F} = base(\rho^n_\mathcal{F}) : base(\gamma^n_\mathcal{F} \wedge^h \theta) \to base(\gamma^{n+1}_\mathcal{F})$ and define $B\mathcal{F}$ to be the telescope of the sequence $\{\cdots \to B\mathcal{F}_n \xrightarrow{r^n_\mathcal{F}} B\mathcal{F}_{n+1} \to \dots\}$.*
By definition of the telescope, we have an inclusion $B\mathcal{F}_n \cong B\mathcal{F}_n \times \{n\} \subset B\mathcal{F}_n \times [n, n+1] \to B\mathcal{F}$.

Definition 5.1.3 *Let α be a $(\mathbb{S}, *)$-fibration $p : Y \to X$. A section of α is a map $s : X \to Y$ with $p \circ s = id_X$. We define the Thom space of α by setting $T(\alpha) := Y/s(X)$ and $T(\alpha) := *$ if $X = \emptyset$.*

Let $\overline{B\mathcal{F}_n}$ be the telescope of the finite sequence $\{B\mathcal{F}_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{n-1}} B\mathcal{F}_n\}$. We can regard $\overline{B\mathcal{F}_n}$ as a CW-subcomplex of $B\mathcal{F}$ and by doing so we have a CW-filtration $\{\overline{B\mathcal{F}_n}\}$ of $B\mathcal{F}$. Since we identify $x \in B\mathcal{F}_i$ in the telescope with its image $r_i(x)$, we have $\overline{B\mathcal{F}_n} \simeq B\mathcal{F}_n$ and there is an universal object $\gamma^n_{\mathcal{F}}$ over $\overline{B\mathcal{F}_n}$.

Definition 5.1.4 *Let X be a CW-complex and $f : X \to B\mathcal{F}$. Moreover, let X^n_f be the maximal CW-subcomplex which is contained in $f^{-1}(\overline{B\mathcal{F}_n})$. This obviously gives a CW-filtration of X with $f(X^n_f) \subset \overline{B\mathcal{F}_n}$. We define $f_n : X^n_f \to \overline{B\mathcal{F}_n}$ by setting $f_n(x) = f(x)$.*

$$\begin{array}{ccc} X & \xrightarrow{f} & B\mathcal{F} \\ \uparrow & & \uparrow \\ X^n_f & \xrightarrow{f_n} & \overline{B\mathcal{F}_n} \end{array}$$

By definition (6.4.3)*, there is an induced fibration $f^*_n\gamma^n_{\mathcal{F}}$ over X^n_f which will be denoted by ζ^n. If $i_n : X^n_f \to X^{n+1}_f$ is the inclusion and ζ^{n+1} the induced fibration over X^{n+1}_f, then clearly $i^*_n\zeta^{n+1} = \zeta^n \oplus \theta$. Together with the maps $s_n : \Sigma T(\zeta^n) = T(\zeta^n \oplus \theta) \to T(\zeta^{n+1})$ we get the Thom spectrum $M(f) = \{T(\zeta^n), s_n\}$.*

Remark: There is a similar construction for a CW-filtration of X with $f(X_n) \subset \overline{B\mathcal{F}_n}$. However, the homotopy type of the Thom spectrum does not depend on the choice of filtration (see [24], IV.5.13).

Example 5.1.5 *The unitary group $U(n)$ acts on $\mathbb{S}^n$ since unitary endomorphisms preserve the norm. Therefore, it is $U(n) \subset \mathcal{F}_n$ which induces a map $f : BU \to B\mathcal{F}$ and thus $MU = M(f)$.*

For a more clear construction of MU have a look at the appendix, section (6.5.3).

Theorem 5.1.6 *The homotopy groups of the spectrum MU are given by $\pi_*(MU) = \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$ and its homology is given by $\pi_*(MU) = \mathbb{Z}[b_1, b_2, \dots], |b_i| = 2i$.*

(See for example [25], pp. 230,231 and p. 399.)

Remark: The ring $\mathbb{Z}[b_1, b_2, \dots]$ with $|b_i| = 2i$ is also known as Lazardring from formal group law theory.

We now introduce the Thom isomorphism corresponding to a map $X \to B\mathcal{F}$ which is an important ingredient to our proof.
Recall that a fibration $F \to E \to B$ is orientable, if $\pi_1(B)$ acts trivially on $H_*(F; G)$.

Theorem 5.1.7 *Let $\alpha = \{p : Y \to X\}$ be an orientable $(\mathbb{S}^n, *)$-fibration and X path-connected, then there are Thom isomorphisms*

$$\phi_G : H_i(X; G) \xrightarrow{\cong} \tilde{H}_{i+n}(T(\alpha); G) \text{ and } \phi^G : H^i(X; G) \xrightarrow{\cong} \tilde{H}^{i+n}(T(\alpha); G)$$

Proof: (Compare [24].) To prove the isomorphism on homology, we consider the Serre spectral sequence for the relative fibration $(Y, s(X)) \to X$. It is

$$E^2_{p,q} \cong H_p(X; H_q(\mathbb{S}^n, *; G)) \cong \begin{cases} H_p(X; G) & q = n, \\ 0 & \text{otherwise.} \end{cases}$$

As there is only one row with nonzero entries ($q = n$), all differentials d^r are zero for $r \geq 2$. Thus, $E^2_{p,q} = E^\infty_{p,q}$. As this spectral sequence converges to $H_*(Y, s(X); G) \cong \tilde{H}_*(T(\alpha); G)$ we have

$$H_p(X; G) \cong E^2_{p,n} \cong E^\infty_{p,n} \cong \tilde{H}_{p+n}(T(\alpha); G).$$

The isomorphism on cohomology follows analogously with the cohomology Serre spectral sequence.

□

Definition 5.1.8 *Let X be a CW-complex. We say that a map $\alpha = \{f : X \to B\mathcal{F}\}$ is regular if $f(X^{(n-2)}) \subset B\mathcal{F}_n$ for every n.*
Given a regular map as above we define $f_n : X^{(n-2)} \to B\mathcal{F}_n$ via $f_n(x) = f(x)$ for every $x \in X^{(n-2)}$ and set

$$\alpha^n := f_n^* \gamma^n_{\mathcal{F}}$$

With this we can write $M(f) = \{T(\alpha^n)\}$. We call α orientable if α^n is for every $n \geq 2$.

Remark: We will later use an equivalent characterization of orientability. For connected X, we have in fact that $\pi_0(M(f)) = \mathbb{Z}$ if α is orientable and $\pi_0(M(f)) = \mathbb{Z}/2\mathbb{Z}$ if α is not (see proposition 5.24 in [24], pp. 262,263). Moreover, orientability is equivalent to the existence of a lift of $f : X \to B\mathcal{F}$ to $BS\mathcal{F}$ which is the classifying space for all spherical fibrations of degree 1:

$$\begin{tikzcd} & BS\mathcal{F} \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, dashed] & B\mathcal{F} \end{tikzcd}$$

We know this from complex vector bundles where orientability is equivalent to the existence of a lift to BSU:

$$\begin{tikzcd} & BSU \arrow[d] \\ X \arrow[r, "f"] \arrow[ur, dashed] & BU \end{tikzcd}$$

Theorem 5.1.9 *Let X be a CW-complex and $\alpha = \{f : X \to B\mathcal{F}\}$. Additionally, let G be an abelian group and suppose that α is orientable. Then there are isomorphisms*

$$\Phi_G : H_i(X; G) \xrightarrow{\cong} \tilde{H}_i(M(f); G) \quad \textit{and} \quad \Phi^G : H^i(X; G) \xrightarrow{\cong} \tilde{H}^i(M(f); G).$$

Proof: (See [24].) Φ_G can be constructed as

$$H_i(X; G) = H_i(X^{(N-2)}; G) \overset{thm(5.1.7)}{\cong} \tilde{H}_{i+N}(T(\alpha^N); G) = \tilde{H}_i(M(f); G),$$

where $i << N$. The cohomology version is analogous.

□

Remark: There is a version of theorem (5.1.7) for a generalized (co-)homology coming from a spectrum E (see [13]) which of course implies a generalized version of theorem (5.1.9). In fact, we will later use this theorem for a homology theory coming from an Eilenberg-MacLane spectrum.

5.2 The Brown-Peterson spectrum

We said that $\pi_*(MU) = MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$. In particular, it is isomorphic to the complex cobordism ring Ω_*^U.

Theorem 5.2.1 $\Omega_*^U \cong \mathbb{Z}[x_2, x_4, x_6, \dots]$ *and x_{2k} may be taken to be the class $[\mathbb{C}P^k]$ if $k = p - 1$ for some prime p.*

(See for example [25], chapter 12.)

If we localize the spectrum MU at a prime p we can find a unique map of ringspectra $\varepsilon : MU_{(p)} \to MU_{(p)}$, such that $\varepsilon^2 = \varepsilon$ and $\varepsilon_* : MU_{(p)*} \to MU_{(p)*}$ is given by

$$\varepsilon_*[\mathbb{C}P^n] = \begin{cases} [\mathbb{C}P^n] & \text{if } n = p^t - 1 \text{ for some integer } t, \\ 0 & \textit{otherwise}. \end{cases}$$

The existence of this idempotent needs of course to be proven (see for example [13]) and in fact, all this goes back to a famous theorem of Quillen in [21].
The image of a multiplicative idempotent in $MU_*(\ )_{(p)}$ is a natural direct summand and so gives rise to a multiplicative generalized (co-)homology theory: For any spectrum E, we define $BP_*(E) \cong \operatorname{im} \varepsilon_*$ and $BP^* \cong \operatorname{im} \varepsilon^*$ with $\varepsilon_* : MU_{(p)*}(E) \to MU_{(p)*}(E)$ and $\varepsilon^* : MU^*_{(p)}(E) \to MU^*_{(p)}(E)$ respectively. The representing spectrum is called Brown-Peterson spectrum and denoted BP.

Theorem 5.2.2 *The Brown-Peterson spectrum BP is a homotopy-commutative and -associative ringspectrum with $H_*(BP) \cong \mathbb{Z}_{(p)}[l_1, l_2, \dots]$, where $|l_i| = 2(p^i - 1)$, and $H^*(BP; \mathbb{F}_p) = A_p/(\beta)$ with A_p being the Steenrod Algebra and β the Bockstein homomorphism.*

The multiplication on BP is induced by the composition

$$BP \wedge BP \to MU_{(p)} \wedge MU_{(p)} \xrightarrow{\mu_{MU}} MU_{(p)} \xrightarrow{\varepsilon} BP.$$

The grading of the generators of $H_*(BP)$ is of course due to the definition of ε. A detailed proof of this theorem may be found in [24], pp. 413-415.

The question of interest is if BP posseses a strictly commutative model. In fact, this question has not been answered yet though there were many attempts. One attempt was to detect BP as the p-localization of the Thom spectrum $M(f)$, associated to some map $f : X \to B\mathcal{F}$. We will now explain this - unfortunately unsuccessful - attempt in detail.

5.3 BP as a multiplicative Thom spectrum

So far, we introduced the classifying space $B\mathcal{F}$ and constructed a Thom spectrum $M(f)$ for each map $f : X \to B\mathcal{F}$.

Remark: $B\mathcal{F}$ is weakly equivalent to an infinite loop space.
Boardman and Vogt showed in [4], that $\mathcal{F}$ is an E_∞-space and that every classifying space of an E_∞-space is again an E_∞-space. As we said before, a space is E_∞ if and only if it is weakly equivalent to an infinite loop space.

Knowing this we can now consider loop maps $f : X \to B\mathcal{F}$ and state the following theorem which actually is a corollary of theorem 7.1 in [15], which is far more general.

Theorem 5.3.1 *Let $X = \Omega^\infty Y$ and $f : X \to B\mathcal{F}$ be an infinite loop map. Then the associated Thom spectrum $M(f)$ is a E_∞-ringspectrum.*

Thus, if we could show that $M(f)$ localizes to BP for some prime p, $p \neq 2$, BP would be an E_∞-ringspectrum as we explained in (1.5.2) and thus strictly commutative. However, we will prove that $M(f)$ does not even localize to BP if f is only a 2-fold loop map.

Proposition 5.3.2 *Let $X = \Omega^2 Y$ and $f : X \to B\mathcal{F}$ be a 2-fold loop map. Then the associated Thom spectrum $M(f)$ does not localize to BP for any prime $p \neq 2$.*

Outline of the proof:
We will prove this proposition by contradiction. That is, we will assume that BP is indeed the localization of the Thom spectrum associated to a 2-fold loop map $X = \Omega^2 Y \to B\mathcal{F}$. By use of the Thom isomorphism and the Eilenberg-Moore spectral sequence we will then show that this implies $H_*(Y; \mathbb{F}_p) = \mathbb{F}_p[y_1, y_2, \dots]$

with $|y_i| = 2p^i$. However, theorem (4.3.1) on the factorization of secondary cohomology operations tells us, that in this case we would have $y_1^p = 0$ which obviously cannot be true since y_1 is a generator in a torsionfree polynomial ring.

Before we start with the actual proof, we collect some helpful tools that will help later on.

Lemma 5.3.3 *If $H_*(X;\mathbb{Z}_{(p)}) \cong H_*(BP)$, then $X = \Omega^2 Y$ is connected and we can assume that ΩY is connected without loss of generality.*

Remark: We need this in order to apply the theorems of chapter 2 which were valid only for connected topological groups.

Proof: Since we know for any space Z that $H_0(Z;R) = \oplus_{\pi_0(Z)} R$, we see that $\Omega^2 Y$ is actually arcwise connected (and thus connected) as $H_0(\Omega^2 Y;\mathbb{Z}_{(p)}) = H_0(X;\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}$. We can assume ΩY to be connected because if it was not, we would get $\Omega(\Omega Y) = \Omega(Y_j)$ for $\Omega Y = \sqcup_i Y_i$ and the basepoint lying in Y_j. $\square$

Lemma 5.3.4 *$H_*(B\Omega^2 Y) \cong H_*(\Omega Y)$ and $H_*(B\Omega Y) \cong H_*(Y)$.*

Proof: Consider the path-loop fibration $\Omega Y \to PY \to Y$ and its corresponding long homotopy sequence:

$$\cdots \to \pi_n(PY) \to \pi_n(Y) \to \pi_{n-1}(\Omega Y) \to \pi_{n-1}(PY) \to \dots$$

As PY is contractible, we have $\pi_i(PY) = 0$ for all i and thus $\pi_n(Y) \cong \pi_{n-1}(\Omega Y)$, for all n.
Now consider the universal bundle $G \to EG \to BG$ for ΩY. Again, we have a long homotopy sequence:

$$\cdots \to \pi_n(E\Omega Y) \to \pi_n(B\Omega Y) \to \pi_{n-1}(\Omega Y) \to \pi_{n-1}(E\Omega Y) \to \dots$$

As $E\Omega Y$ is as well contractible, it is $\pi_n(B\Omega Y) \cong \pi_{n-1}(\Omega Y)$, for all n. This finally gives $\pi_*(B\Omega Y) \cong \pi_*(Y)$. By replacing ΩY by $\Omega^2 Y$ and Y by ΩY we get as well $\pi_*(B\Omega^2 Y) \cong \pi_*(\Omega Y)$.
We want now apply Whitehead's theorem in order to prove the desired isomorphism on homology. This theorem states, that if there is a map between arcwise connected spaces of the homotopy type of a CW-complex and the induced map on homotopy is an isomorphism, so is the induced map on homology. Thus, we have to show that $B\Omega Y, Y$ and $B\Omega^2 Y, \Omega Y$ respectively are arcwise connected (i.e. $\pi_0 = *$). In the case of the classifying spaces this is obvious. In the case of Y and ΩY we can assume them to be arcwise connected with a similar argument as above. If they were not the loop space on them would be the loop space on the path-component of the basepoint so we can restrict to this component.

□

Proof of (5.3.2)**:** Suppose $M(f)$ localizes to BP for any prime $p \neq 2$. Then $\pi_0(M(f)_{(p)}) = \mathbb{Z}_{(p)}$ and thus $f : X \to B\mathcal{F}$ is orientable. Therefore, we can apply the $H\mathbb{Z}_{(p)}$-Thom isomorphism which gives

$$H\mathbb{Z}_{(p)}{}_*(X) = H_*(X; \mathbb{Z}_{(p)}) \cong H\mathbb{Z}_{(p)}{}_*(M(f)) = H_*(BP) = \mathbb{Z}_{(p)}[l_1, l_2, ...]$$

with $|l_i| = 2p^i - 2$.

As we want to use the Eilenberg-Moore spectral sequence in the following, we have to change the coefficients of our homology. The Eilenberg-Moore spectral sequence was only constructed for coefficients in a field, thus we exchange $\mathbb{Z}_{(p)}$-coefficients for $\mathbb{F}_p$-coefficients. We have to do this exchange of coefficients anyway, as we want to use secondary cohomology operations later on and these were only defined for cohomology with $\mathbb{F}_p$-coefficients.
As $H_*(X; \mathbb{Z}_{(p)})$ is a free polynomial algebra and the exchange of coefficients does not affect the generators, we get $H_*(X, \mathbb{F}_p) = \mathbb{F}_p[l_1, l_2, \ldots]$.

By use of the Eilenberg-Moore spectral sequence, we want to show that $H^*(Y; \mathbb{F}_p) = \mathbb{F}_p[y_1, y_2, \ldots]$ with $|y_k| = 2p^k$. As a first step, we apply proposition (2.1.5) on $X = \Omega^2 Y$ in order to calculate $H_*(B\Omega^2 Y; \mathbb{F}_p) \cong H_*(\Omega Y; \mathbb{F}_p)$.

Calculation of $H_*(\Omega Y)$
Understanding ΩY as $B\Omega^2 Y = BX$, we use proposition (2.1.4) in order to calculate $H_*(\Omega Y)$.
For this purpose, we first have to determine $E^2_{**} \cong \mathrm{Tor}^{H_*(X;\mathbb{F}_p)}_{**}(\mathbb{F}_p, \mathbb{F}_p)$.
Recall that $H_*(X; \mathbb{F}_p) \cong H_*(BP; \mathbb{F}_p) = \mathbb{F}_p[l_1, l_2, \ldots]$ with $|l_i| = 2(p^i - 1)$. For simplification, we first calculate $\mathrm{Tor}^{\mathbb{F}_p[l_1,\ldots,l_k]}_*(\mathbb{F}_p, \mathbb{F}_p)$ with l_i graded as above. The generalisation to infinitely many generators will follow afterwards.
Since $\mathbb{F}_p[l_1, \ldots, l_k]$ and $\mathbb{F}_p$ are graded, we can choose our resolution in the category of graded $\mathbb{F}_p[l_1, \ldots, l_k]$-modules. (We need here that this category has enough free objects. Fortunately, this is true.) Thus, there will be an induced grading of $\mathrm{Tor}^{\mathbb{F}_p[l_1,\ldots,l_k]}_n(\mathbb{F}_p, \mathbb{F}_p)$.
One main point is how we understand $\mathbb{F}_p$ as $\mathbb{F}_p[l_1, \ldots, l_k]$-module. The module structure of a graded module M over a graded ring A has to fulfill the rule $A_i M_j \subset M_{i+j}$. In our case, $\mathbb{F}_p$ only lives in degree zero. That is, we have no choice but to demand that $(\mathbb{F}_p[l_1, \ldots, l_k])_0 = \mathbb{F}_p$ acts on $\mathbb{F}_p$ by multiplication and elements of higher degree go to zero. Moreover, this makes sense in a more intuitive way as we can understand $\mathbb{F}_p$ as $\mathbb{F}_p[l_1, \ldots, l_k]/(l_1, \ldots, l_k)$.

Let us define $R := \mathbb{F}_p[l_1, \ldots, l_k]$ and $I := (l_1, \ldots, l_k)$ in order to keep things a little bit clearer. Then let $l : R^k \to R$ be the linear form given by $l(r_1, \ldots, r_k) = \sum l_i r_i$.

This gives rise to a complex $K(l) = (\Lambda_R^* R^k, d_l)$, called the Koszul complex, where $\Lambda_R^* R^k$ denotes the graded exterior algebra over R^k. The map $d_l : \Lambda_R^{n+1} R^k \to \Lambda_R^n R^k$ is given by

$$d_l(v_0 \otimes \cdots \otimes v_n) = \sum_{i=0}^{n} (-1)^i l(v_i) v_0 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n,$$

where $v_i \in R^k$. Obviously, we have $d_l = l$ for $n = 0$. To see that $K(l)$ is indeed a complex, we have to check that $d_l^2 = 0$. We do an example first in order to understand what is going on.
Let $n = 2$, then:

$$\begin{aligned} d_l^2(v_0 \otimes v_1 \otimes v_2) &= d_l(l(v_0)v_1 \otimes v_2 - l(v_1)v_0 \otimes v_2 + l(v_2)v_0 \otimes v_1) \\ &= l(v_0)d_l(v_1 \otimes v_2) - l(v_1)d_l(v_0 \otimes v_2) + l(v_2)d_l(v_0 \otimes v_1) \\ &= l(v_0)l(v_1)v_2 - l(v_0)l(v_2)v_1 - (l(v_1)l(v_0)v_2 - l(v_1)l(v_2)v_0) \\ &\quad + l(v_2)l(v_0)v_1 - l(v_2)l(v_1)v_0 \\ &= 0 \end{aligned}$$

since R is commutative. We see that when applying d_l the second time, the positions are changed. In the first summand for example, v_1 goes to the 0th position and v_2 to the first, because v_0 is missing. This causes opposite signs and thus the terms add up to zero. Hence:

$$\begin{aligned} d_l^2(v_0 \otimes \cdots \otimes v_n) &= d_l(\sum_{i=0}^{n} (-1)^i l(v_i) v_0 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n) \\ &= \sum_{i=0}^{n} (-1)^i l(v_i) d_l(v_0 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n) \\ &= \sum_{i=0}^{n} (-1)^i l(v_i) (\sum_{j=0}^{i-1} (-1)^j l(v_j) v_0 \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n \\ &\quad + \sum_{j=i+1}^{n} (-1)^{j-1} l(v_j) v_0 \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n) \\ &= 0 \end{aligned}$$

We claim now, that $K(l)$ is a resolution of $R/I \cong \mathbb{F}_p$ over R. This means, the following sequence is exact:

$$\cdots \to \Lambda_R^{n+1} R^k \xrightarrow{d_l} \Lambda_R^n R^k \xrightarrow{d_l} \cdots \xrightarrow{d_l} \Lambda_R^1 R^k \xrightarrow{l} R \to R/I \to 0$$

This is equivalent to $H_n(K(l)) = 0$ if $n > 0$ and $H_0(K(l)) = R/(l_1, \ldots, l_k)$. We proof this by induction on k.
If $k = 1$, then $I = (l_1)$ and $K(l) = K(l_1)$ is the complex

$$0 \to R \xrightarrow{l_1} R \to 0.$$

Multiplication with l_1 is injective, thus $H_n(K(l_1)) = 0$ for $n > 0$ and $H_0(K(l_1)) = R/l_1R = R/I$.
Suppose now that $K(l)$ is a resolution as stated for $k-1$. Consider the length-one-complex $K(l_k)$ and the exact sequence of complexes $0 \to K_0 \to K(l_k) \to K_1 \to 0$ which looks like this:

$$
\begin{tikzcd}
\textit{degree} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
1 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r, "id"] \arrow[d, "\cdot l_k"] & R \arrow[r] \arrow[d, "\cdot 0"] & 0 \arrow[d] \\
0 & 0 \arrow[r] \arrow[d] & R \arrow[r, "id"] \arrow[d] & R \arrow[r, "\cdot 0"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

Tensoring over R with the Koszul complex $K = K(l_1, \dots, l_{k-1})$ maintains exactness. Moreover, the middle term is $K(l_k) \otimes_R K = K(l)$. The short exact sequence $0 \to K_0 \otimes K \to K(l) \to K_1 \otimes K \to 0$ gives rise to a long exact homology sequence:

$$\cdots \to H_{n+1}(K_1 \otimes_R K) \xrightarrow{\delta} H_n(K_0 \otimes_R K) \to H_n(K(l)) \to H_n(K_1 \otimes_R K) \xrightarrow{\delta} \dots$$

By application of the Künneth formula, this gives

$$\cdots \to R \otimes_R H_n(K) \xrightarrow{\delta} R \otimes_R H_n(K) \to H_n(K(l)) \to R \otimes_R H_{n-1}(K) \xrightarrow{\delta} \dots$$

Thus, δ is simply multiplication by l_k because $H_n(K(l)) = H_n(K(l_k) \otimes_R K) = R/l_kR \otimes_R H_n(K)$ (again by use of the Künneth formula). The exactness of the above sequence is equivalent to the exactness of the following short sequence:

$$0 \to \operatorname{Coker}(H_n(K) \xrightarrow{l_k} H_n(K)) \to H_n(K(l)) \to \operatorname{Ker}(H_{n-1}(K) \xrightarrow{l_k} H_{n-1}(K)) \to 0$$

The inductive hypothesis is $H_n(K) = 0$ for $n > 0$ and $H_0(K) = R/(l_1R + \dots + l_{k-1}R)$, thus $H_n(K(l)) = 0$ for $n > 1$.
For $n = 1$, $H_1(K(l)) = \operatorname{Ker}(H_0(K) \xrightarrow{l_k} H_0(K))$. As this multiplication is injective, we have $H_1(K(l)) = 0$.
For $n = 0$, $H_0(K(l)) = \operatorname{Coker}(H_0(K) \xrightarrow{l_k} H_0(K)) = R/I$ as we wanted to show.

So far we have that $K(l)$ is a free resolution of R/I over R. In order to calculate $\operatorname{Tor}_*^R(R/I, R/I)$, we have to determine the homology of $(\Lambda_R^*(R^k) \otimes_R R/I, d_l \otimes 1)$. By definition, the image of l is in I, thus $d_l \otimes 1 = 0$ because of the module structure of R/I over R. Therefore, $\operatorname{Tor}_*^R(R/I, R/I) = \Lambda_R^*(R^k) \otimes_R R/I = \Lambda_{R/I}^*((R/I)^k)$. Moreover, $(R/I)^k = (\mathbb{F}_p)^k \cong I/I^2$ by counting generators. Hence,

$\mathrm{Tor}_*^R(R/I,R/I) \cong \Lambda^*_{R/I}(I/I^2) = \Lambda^*_{\mathbb{F}_p}(l_1,\dots,l_k)$.
Our last task here is to check that the canonical product on the Tor-groups is identical to the exterior algebra product. Hence, consider $f : K(l) \otimes_R K(l) \to K(l)$ where $(K(l) \otimes_R K(l))_n = \oplus_{p+q=n} K(l)_p \otimes_R K(l)_q$. This is in fact a homomorphism of complexes lifting the muliplication on R/I such that each f_n is the multiplication in the exterior algebra. Since all differentials are zero, one sees very easily by having a look on the product formula of the Tor-term (section (2.1)) that both products correspond.

The ideas presented so far do hold of course in a more general context, too. In fact, the presentation was guided by the general proof of Loday in [17] (pp. 103-105) and supplemented by details.

We now have to generalize the above result to infinitely many generators. Consider the direct system $\cdots \to R^k \xrightarrow{j} R^{k+1} \to \dots$ with j being the inclusion $(r_1,\dots,r_k) \mapsto (r_1,\dots,r_k,0)$ and its direct limit R^∞. Then the following is a direct system

$$\begin{tikzcd}
 & \cdots \arrow[d, "j^*"] & \cdots \arrow[d, "j^*"] & \cdots & \cdots \arrow[d, "j^*"] & \cdots \arrow[d, "j^*"] & \\
\cdots \arrow[r] & \Lambda^{n+1}_R R^k \arrow[r, "d_{l(k)}"] \arrow[d, "j^*"] & \Lambda^n_R R^k \arrow[r, "d_{l(k)}"] \arrow[d, "j^*"] & \cdots \arrow[r, "d_{l(k)}"] & \Lambda^0_R R^k = R \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \Lambda^{n+1}_R R^{k+1} \arrow[r, "d_{l(k+1)}"] \arrow[d, "j^*"] & \Lambda^n_R R^{k+1} \arrow[r, "d_{l(k+1)}"] \arrow[d, "j^*"] & \cdots \arrow[r, "d_{l(k+1)}"] & \Lambda^0_R R^{k+1} = R \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d] & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

with direct limit

$$\cdots \longrightarrow \Lambda^{n+1}_R R^\infty \xrightarrow{d_{l(\infty)}} \Lambda^n_R R^\infty \xrightarrow{d_{l(\infty)}} \cdots \xrightarrow{d_{l(\infty)}} \Lambda^0_R R^\infty = R \longrightarrow R/I \longrightarrow 0$$

Thereby, $l(k)$ is the linear form belonging to $(l_1,\dots,l_k)$ and $l(\infty)$ the one belonging to $(l_1,l_2,\dots)$ with $|l_i| = 2p^i - 2$. The latter is well-defined since the elements of the direct limit R^∞ do only have finitely many non-zero components. By (6.3.2), this direct limit is a resolution of $R/I = \mathbb{F}_p[l_1,l_2,\dots]/(l_1,l_2,\dots) = \mathbb{F}_p$. If we tensor with R/I and drop the last term, the above (without the last terms) is still a direct system with direct limit

$$\cdots \longrightarrow \Lambda^{n+1}_R R^\infty \otimes_R R/I \xrightarrow{d_{l(\infty)}\otimes 1} \Lambda^n_R R^\infty \otimes_R R/I \xrightarrow{d_{l(\infty)}\otimes 1} \cdots \xrightarrow{d_{l(\infty)}\otimes 1} \Lambda^0_R R^\infty \otimes_R R/I \longrightarrow 0$$

The homology groups of this complex are $\mathrm{Tor}_*^R(R/I,R/I)$. By (6.3.2), the homology of the direct limit is the direct limit of the homology of the underlying direct system. Thus we finally get:

$$\mathrm{Tor}_*^{\mathbb{F}_p[l_1,l_2,\dots]}(\mathbb{F}_p,\mathbb{F}_p) = \varinjlim \Lambda^*_{\mathbb{F}_p}(l_1,\dots,l_k) = \Lambda^*_{\mathbb{F}_p}(l_1,l_2,\dots)$$

$\mathrm{Tor}_n^{\mathbb{F}_p[l_1,l_2,\dots]}(\mathbb{F}_p,\mathbb{F}_p) = \Lambda^n_{\mathbb{F}_p}(l_1,l_2,\dots)$ has an internal grading because of the grading of the l_i. Thus, $E^2_{**} \cong \Lambda^{**}_{\mathbb{F}_p}(l_1,l_2,\dots)$, that is, the E_2-page looks like this:

$$\begin{array}{r|lllll}
q & & & & & \\
\dots & & \dots & \dots & \dots & \\
2(p+p^2+p^3)-6 & & & & l_1l_2l_3 & \\
2p^3-2 & & l_3 & & & \\
2(p+p^2)-4 & & & l_1l_2 & & \\
2p^2-2 & & l_2 & & & \\
2p-2 & & l_1 & & & \\
 & \mathbb{Z}_{(p)} & & & & \\
\hline
(0,0) & & 1 & 2 & 3 & \dots \quad p
\end{array}$$

Where there is no entry we mean of course 0.
Let us consider the related maps $d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$. It is obvious, that they either start in zero or go to zero. The same holds for all other differentials. To see this, consider the total degree $(p+q)$ of elements. In the first column, the total degree equals $2p^i-1$ for some i. In the second one, it equals $2(p^i+p^j)-2$ and in the third $2(p^i+p^j+p^k)-3$. In general, the total degree equals $2(\sum_{i=1}^{\infty} a_ip^i) - \#\{a_i \neq 0\}, a_i \in \{0,1\}$. The differentials have total degree -1. Hence, if we can show, that there exist no elements whose total degree differs by 1, then all differentials must be zero. If the total degree of two elements differs by 1, then one of them has to be even and one odd. Thus, the number of $a_i \neq 0$ must differ by an odd amount, at least one. However, this means that the total degrees differ by at least one p^i-summand which is obviously larger than 1. Consequently, all differentials are zero and the spectral sequence collapses in E^2, that is $E^2 = E^\infty$.
By lemma (2.0.14), our spectral sequence converges to total$(E^\infty_{*,*}) \cong \Lambda_{\mathbb{F}_p}(z_1,z_2,\dots)$ with $|z_i| = 2p^i-1$.
Thus, $H_*(\Omega Y;\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(z_1,z_2,\dots), |z_i| = 2p^i-1$. Applying theorem (2.1.5) on

ΩY then gives $H^*(B\Omega Y;\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p) \cong \mathbb{F}_p[y_1, y_2, \dots]$ with $|y_k| = 2p^k$ as wanted.

Finally, we now take advantage of cohomology operations in order to produce a contradiction: On the one hand, we have $\mathcal{P}^p(y_1) = y_1^p$ due to $|y_1| = 2p$ and $y_1^p \neq 0$ since y_1 is a generator in the torsion free polynomial ring $\mathbb{F}_p[y_1, y_2, \dots]$. On the other hand, the factorization of $\mathcal{P}^p$ by secondary cohomology operations tells us that $\mathcal{P}^p(y_1) = \nu_0^{-1}(b_0\mathcal{R} + \sum_\gamma c_{0,\gamma}\Gamma_\gamma)$. However, the operations Γ_γ are of odd degree. Thus, $\Gamma_\gamma = 0$ since our polynomial algebra only lives in even degree. Moreover, the operation $\mathcal{R}$ is of degree $4(p-1)$. As $2p^i + 4(p-1) \neq 2p^j$ for all i, j and $p \neq 2$, it is zero as well. Consequently, $\mathcal{P}^p(y_1) = 0$. $\lightning$

$\square$

Final words

We proved that BP is not the localization of a Thom spectrum associated to a 2-fold loop map and thus cannot be understood as a Thom spectrum in this way. So, what does this tell us concerning our question if there is a strictly commutative product structure on BP? Unfortunately, this does not tell us very much. For example, the above proof would hold as well for $BP\langle 1\rangle$ which is the spectrum representing $BP\langle 1\rangle_* = \mathbb{Z}_{(p)}[l_1], |l_1| = 2p-2$, since the degree of this first generator was exactly what caused the contradiction. However, it is known that $BP\langle 1\rangle$ does have a strictly commutative product structure (see[3]). Moreover, the proof obviously holds for every $BP\langle n\rangle$ representing $BP\langle n\rangle_* = \mathbb{Z}_p[l_1, \dots, l_n], |l_i| = 2(p^i - 1)$, but we do not yet know if they possess a strictly commutative model.

Chapter 6

Appendix

6.1 Axioms for a reduced homology theory

Definition 6.1.1 *Let $\mathcal{PT}'$ be the category of pointed topological spaces and homotopy classes of basepoint preserving maps. We denote by Σ the (reduced) suspension functor defined by $\Sigma(X, x_0) = (\Sigma X, *)$ and $\Sigma[f] = [1_{\mathbb{S}^1} \wedge f]$ for $(X, x_0) \in \mathcal{PT}'$. A reduced homology theory $\tilde{h}_*$ on $\mathcal{PT}'$ is a collection of covariant functors $\tilde{h}_n$ from $\mathcal{PT}'$ to the category of (abelian) groups and natural equivalences $\sigma_n : \tilde{h}_n \to \tilde{h}_{n+1} \circ \Sigma$ satisfying the exactness axiom:*
For every pointed pair (X, A, x_0) with inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0) \hookrightarrow (X \cup_i CA, *)$ the sequence

$$\tilde{h}_n(A, x_0) \xrightarrow{i_*} \tilde{h}_n(X, x_0) \xrightarrow{j_*} \tilde{h}_n(X \cup_i CA, *)$$

is exact.
By saying that σ_n is a natural equivalence, we mean that the diagram

$$\begin{tikzcd} \tilde{h}_n(X) \arrow[r, "\sigma_n"] \arrow[d] & \tilde{h}_{n+1}(\Sigma X) \arrow[d] \\ \tilde{h}_n(Y) \arrow[r, "\sigma_n"] & \tilde{h}_{n+1}(\Sigma Y) \end{tikzcd}$$

commutes.

Many of the readers might know about reduced singular homology and its nice property of being zero on any single point $\{x\}$. This follows for every reduced homology theory from the exactness axiom by considering the inclusions $i : (\{x\}, x) \to (\{x\}, x)$, and $j : (\{x\}, x) \to (\{x\} \cup_i C'\{x\}, *)$, where C' is the reduced cone: Both inclusions are in fact the identity and the sequence

$$\tilde{h}(\{x\}) \xrightarrow{id} \tilde{h}(\{x\}) \xrightarrow{id} \tilde{h}(\{x\})$$

is exact only if $\tilde{h}(\{x\}, x) = 0$ (compare [25], p.110).

There is another axiom which will be of some importance.

Definition 6.1.2 (Wedge axiom) *A reduced homology theory satisfies the wedge axiom, if for every collection $(X_\alpha, x_\alpha)_{\alpha\in A}$ of pointed spaces the inclusions $i_\alpha : X_\alpha \to \bigvee_{\beta\in A} X_\beta$ induce an isomorphism*

$$i_{\alpha *} : \oplus_{\alpha\in A}\tilde{h}_n(X_\alpha) \to \tilde{h}_n(\bigvee_{\alpha\in A} X_\alpha)$$

for all n.

The dual notion of a reduced homology theory is a reduced cohomology theory. Its axioms are essentially the same except for it being a contravariant functor, that is all the arrows involved go in the opposite direction.

6.2 Hopf algebras

Definition 6.2.1 *A bialgebra over k is an k-algebra H with product $\times$, together with algebra homomorphisms Δ and ε making H into a coalgebra. We call H Hopf algebra if in addition there is a k-module homomorphism $s : H \to H$ such that the following diagrams commute:*

$$\begin{tikzcd} H\otimes H \arrow[rr, "s\otimes id_H"] & & H\otimes H \arrow[d, "\times"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & k \arrow[r] & H \end{tikzcd} \qquad \begin{tikzcd} H\otimes H \arrow[rr, "id_H\otimes s"] & & H\otimes H \arrow[d, "\times"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & k \arrow[r] & H \end{tikzcd}$$

The homomorphism s is called *antipode* and evidently reminds a little bit of an inversion.

The easiest example of a Hopf algebra is the polynomial algebra $R[\alpha]$. Its coproduct must be given by $\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha$ since there are no elements of lower - but nonzero - degree than α.
Further examples are exterior and divided power algebras.

An important fact about Hopf algebra is the relationship between the (co-)product structure of a graded Hopf algebra A_* and its graded dual A^* with $A^k = Hom(A_k, R)$: The coproduct of A_* determines the product of A^* and vice versa.
In the case of a divided power algebra the coproduct fulfills $\Delta(\gamma_k(\alpha)) = \sum_i \gamma_i(\alpha)\otimes\gamma_{k-i}(\alpha)$. Thus, Δ_{ij} takes $\gamma_{i+j}(\alpha)$ to $\gamma_i(\alpha)\otimes\gamma_j(\alpha)$. So if x_i is a

generator of A^* that is dual to $\gamma_i(\alpha)$ then $x_i x_j = x_{i+j}$ which is the product formula in a polynomial algebra.
Conversely, the coproduct in a polynomial algebra $R[x]$ fulfills $\Delta(x^n) = (x \otimes 1 + 1 \otimes x)^n = \sum_i \binom{n}{i} x^i \otimes x^{n-i}$ if x. So if α_i is dual to x_i, the product in the dual algebra has to fulfill $\alpha_i \alpha_{n-i} = \binom{n}{i} \alpha_n$ which is exactly the multiplication in a divided power algebra.
Thus, $\Gamma_R(\alpha)$ and $R[x]$ are dual to each other as Hopf-algebras.

6.3 Direct limits of chain complexes

Definition 6.3.1 *Let R be a ring and $C_*[n]$ a chain complex of R-modules for each n. A direct system of chain complexes $(C_*[n], f_*[n])$ is a sequence of chain maps*

$$C_*[0] \xrightarrow{f_*[0]} C_*[1] \xrightarrow{f_*[1]} C_*[2] \xrightarrow{f_*[2]} \cdots$$

Its direct limit is a chain complex of R-modules $\varinjlim(C_*[n], f_*[n])$ together with maps $\phi_n : C_*[n] \to \varinjlim(C_*[n], f_*[n])$ such that $\phi_{n+1} \circ f_*[n] = \phi_n$ which fulfills the following universal property:
For each chain complex C'_* together with chain maps $\psi_n : C_*[n] \to C'_*$ satisfying $\psi_{n+1} \circ f_*[n] = \psi_n$ for all $n \geq 0$, there exists a unique chain map $\psi : \varinjlim(C_*[n], f_*[n]) \to C'_*$ such that $\psi \circ \phi_n = \psi_n$ for all $n \geq 0$.

$$\begin{tikzcd}
C_*[n] \arrow[dd, "f_*[n]"] \arrow[dr, "\phi_n"] \arrow[drr, "\psi_n"] & & \\
& \varinjlim(C_*[n], f_*[n]) \arrow[r, dashed, "\exists!\psi"] & C'_* \\
C_*[n+1] \arrow[ur, "\phi_{n+1}"] \arrow[urr, "\psi_{n+1}"'] & &
\end{tikzcd}$$

If $(C_*[n], f_*[n])$ is a direct system of chain complexes, then the m-th R-chain module of its direct limit is equal to the direct limit $\varinjlim(C_m[n])$ of the direct system of R-modules $(C_m[n], f_m[n])$. With this, we can state the following lemma saying that the direct limit exchanges with homology.

Lemma 6.3.2 *Let $(C_*[n], f_*[n])$ be a direct system of chain complexes over R. Then the map*

$$\varinjlim H_m(C_*[n]) \to H_m(\varinjlim C_*[n])$$

is bijective for all $m \in \mathbb{Z}$.

The proof of this lemma, needs the following one:

Lemma 6.3.3 *Let R be a ring and M_n, N_n, P_n R-modules for all n. We call a short sequence of direct systems of R-modules exact, if for each $n \geq 0$ the corresponding sequence of R-modules is exact. For each short exact sequence of direct systems*

$$0 \to (M_n, f_n) \to (N_n, g_n) \to (P_n, h_n) \to 0$$

we get an induced short sequence of R-modules

$$0 \to \varinjlim(M_n, f_n) \to \varinjlim(N_n, g_n) \to \varinjlim(P_n, h_n) \to 0$$

that is again exact.

Both proofs may be found in [18] (p. 114 and p. 118). The exactness in lemma (6.3.3) is proven by really looking at elements. This is not difficult but a little bit technical. Lemma (6.3.2) is proven by a sharp look at the following short exact sequences

$$\begin{aligned} 0 \to B_m[n] \xrightarrow{i_m[n]} Z_m[n] \xrightarrow{p_m[n]} H_m(C_*[n]) \to 0, \\ 0 \to Z_m[n] \xrightarrow{j_m[n]} C_m[n] \xrightarrow{c_m[n]} B_{m-1}[n] \to 0, \\ 0 \to B_m[n] \xrightarrow{k_m[n]} C_m[n] \longrightarrow \operatorname{coker}(k_m[n]) \to 0, \end{aligned}$$

where $B_m[n]$ denotes the image of $c_{m+1}[n] : C_{m+1}[n] \to C_m[n]$ and $Z_m[n]$ denotes the kernel of $c_m[n] : C_m[n] \to C_{m-1}[n]$, and use of lemma (6.3.3).

6.4 Bundles and fibrations

In the following, all maps are assumed to be continous.

Our (very short) presentation restricts to complex vector bundles, since they are the ones we need for the concrete construction of MU. The constructions we make hold as well for real vector bundles of course.

Definition 6.4.1 *A complex vector bundle ξ of dimension n over a space B is a bundle over B such that each point $b \in B$ has a neighborhood U and a homeomorphism $h_U : p^{-1}(U) \to U \times \mathbb{C}^n$ such that $p_1 \circ h_U = p|_{p^{-1}(U)}$ where p_1 denotes projection onto the first factor U.*

Example 6.4.2 *The easiest example is the trivial vector bundle over X with total space $E = X \times \mathbb{C}^n$ and $p : E \to X$ being the projection onto the first factor. The trivial vector bundle for $n = 1$ is called trivial line bundle.*

Definition 6.4.3 *If $p : E \to B$ is a (complex) vector bundle ξ over B and $f : X \to B$ is continuous, then the induced bundle $f^*(\xi)$ over X is the one with total space $f^*(E) = \{(x, e) \in X \times E : f(x) = p(e)\}$ and projection onto the first component. In other words, $f^*(E)$ is the pullback of f and p.*

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

Definition 6.4.4 *Let ξ_1 and ξ_2 be (complex) vector bundles of dimensions n_1 and n_2 over spaces X_1 and X_2 with total spaces E_1 and E_2. Then their external sum $\xi_1 \times \xi_2$ is a bundle of dimension $n_1 + n_2$ over $X_1 \times X_2$ given by the map $p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$.*
When $X_1 = X_2 = X$, the Whitney sum $\xi_1 \oplus \xi_2$ is the bundle over X induced from $\xi_1 \times \xi_2$ by the diagonal map $\Delta : X \to X \times X$.

$$\begin{tikzcd} \Delta^*(X \times X) \arrow[r] \arrow[d] & E_1 \times E_2 \arrow[d, "p_1 \times p_2"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

Proposition 6.4.5 *Let ξ_1, ξ_2 be complex vector bundles over X_1, X_2 respectively, then there is a natural homeomorphism*

$$T(\xi_1 \times \xi_2) \xrightarrow{\cong} T(\xi_1) \wedge T(\xi_2)$$

Proof: (See [25].) We know that $\mathbb{D}^n \times \mathbb{D}^m \cong \mathbb{D}^{n+m}$. The fibre of $\xi_1 \times \xi_2$ (see (6.4.4)) over $(x, y) \in X \times Y$ is ${\xi_1}_x \times {\xi_2}_y$ and we have a homeomorphism $\mathbb{D}(\xi_1)_x \times \mathbb{D}(\xi_2)_y \cong \mathbb{D}(\xi_1 \times \xi_2)_{(x,y)}$ for all (x, y). This gives a homeomorphism $\mathbb{D}(\xi_1) \times \mathbb{D}(\xi_2) \cong \mathbb{D}(\xi_1 \times \xi_2)$ so that $\mathbb{D}(\xi_1) \times \mathbb{S}(\xi_2) \cup \mathbb{S}(\xi_1) \times \mathbb{D}(\xi_2)$ is mapped onto $\mathbb{S}(\xi_1 \times \xi_2)$. Thus

$$\begin{aligned} T(\xi_1) \times T(\xi_2) &= \mathbb{D}(\xi_1)/\mathbb{S}(\xi_1) \wedge \mathbb{D}(\xi_2)/\mathbb{S}(\xi_2) \\ &\cong \frac{\mathbb{D}(\xi_1)/\mathbb{S}(\xi_1) \times \mathbb{D}(\xi_2)/\mathbb{S}(\xi_2)}{\mathbb{D}(\xi_1) \times \mathbb{S}(\xi_2) \cup \mathbb{S}(\xi_1) \times \mathbb{D}(\xi_2)} \\ &\cong \frac{\mathbb{D}(\xi_1) \times \mathbb{D}(\xi_2)}{\mathbb{D}(\xi_1) \times \mathbb{S}(\xi_2) \cup \mathbb{S}(\xi_1) \times \mathbb{D}(\xi_2)} \\ &\cong \mathbb{D}(\xi_1 \times \xi_2)/\mathbb{S}(\xi_1 \times \xi_2) = T(\xi_1 \times \xi_2) \end{aligned}$$

□

Corollary 6.4.6 *If the complex vector bundle ξ is isomorphic to the Whitney sum $\xi' \oplus \varepsilon$, where ε denotes the trivial line bundle, then $T(\xi) = \Sigma^2 T(\xi')$.*

Proof: (Compare [25].) We have $T(\varepsilon) = \mathbb{D}^2/\mathbb{S}^1 = \mathbb{S}^2$. We can regard $\xi' \oplus \varepsilon$ over X as $\xi' \times \varepsilon$ over $X \times X$. Hence, with proposition (6.4.5) we get

$$T(\xi' \oplus \varepsilon) = T(\xi' \times \varepsilon) \cong T(\xi') \wedge T(\varepsilon) \cong T(\xi') \wedge \mathbb{S}^2 \cong \Sigma^2 T(\xi').$$

□

Let us turn to fibrations now.

Definition 6.4.7 *A fibration is a bundle $p : E \to B$ that satisfies the following homotopy lifting property:*
For every map $F : X \times I \to B$ and every $g : X \to E$ with $p \circ g(x) = F(x, 0)$, there exists $G : X \times I \to E$ with $G(x, 0) = g(x)$ and $p \circ G = F$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed, "\exists G"] & B \end{tikzcd}$$

where i_0 maps $x \in X$ onto $(x, 0) \in X \times I$.

Definition 6.4.8 *Let $p : E \to B$ be a bundle. A section is a map $s : B \to E$ such that $p \circ s = id_B$. A pair (ξ, s_ξ) consisting of a bundle ξ and a corresponding section s_ξ is called a sectioned bundle.*

Definition 6.4.9 *Given two sectioned bundles $(\xi, s_\xi), (\eta, s_\eta)$, a sectioned bundle morphism is bundle morphism $\phi : \xi \to \eta$ which respects the sections, i.e. $\phi|total(\xi) \circ s_\xi = s_\eta \circ \phi|base(\xi)$.*

Definition 6.4.10 *A sectioned fibration is a sectioned bundle (ξ, s_ξ) such that ξ is a fibration and $\hat{s_\xi} : id_{base(\xi)} \subset \xi$ is a cofibration over the basespace of ξ.*

Definition 6.4.11 *A $(F, *)$-fibration is a sectioned fibration (ξ, s_ξ) such that $(F_x, s_\xi(x))$ is homotopy equivalent to $(F, *)$ with respect to basepoints for every $x \in base(\xi)$.*
A $(F, *)$-morphism is a sectioned bundle morphism $\phi = (g, f) : (\xi, s_\xi) \to (\eta, s_\eta)$ such that g respects fibers and sections with respect to f, that is

$$g|F_x : F_x \to F_{f(x)} \text{ and } g|F_x : (F_x, s_\xi(x)) \to (F_{f(x)}, s_\eta(f(x)),$$

are (pointed) homotopy equivalences for every $x \in base(\xi)$.

Definition 6.4.12 *A universal $(F, *)$-fibration is an $(F, *)$-fibration $\gamma^F = \{p_F : E_F \to B_F\}$ with the following properties:*
(1) Every $(F, *)$-fibration over a CW-space X is equivalent to a fibration $f^*\gamma^F$ for some $f : X \to B_F$.

(2) *Let $f, g : X \to B_F$ be two maps of a CW-space X. Then the $(F,*)$-fibrations $f^*\gamma^F$ and $f^*\gamma^F$ are equivalent if and only if $f \simeq g$.*
The basespace B_F of a universal $(F,*)$-fibration is called a classifying space for $(F,*)$-fibrations. If an $(F,*)$-fibration ξ is equivalent to $f^*\gamma^F$ for some $f : base(\xi) \to B_F$, we say that f classifies ξ or that f is a classifying map for ξ.
A classifying morphism for an $(F,*)$-fibration ξ is any $(F,*)$-morphism $\phi : \xi \to \gamma^F$.

6.5 The Thom spectrum MU

The construction we present here is mainly guided by Ravenel's exposition in [22].

Definition 6.5.1 (Thom space) *Given a complex vector bundle $\xi = \{p : E \to B\}$ with a Hermitian metric, the disk bundle $\mathbb{D}(\xi)$ consists of all vectors $v \in E$ with $|v| \leq 1$ and the sphere bundle $\mathbb{S}(\xi)$ consists of all vectors $v \in E$ with $|v| = 1$. We define the Thom space $T(\xi)$ to be the quotient $\mathbb{D}(\xi)/\mathbb{S}(\xi)$.*
A map $f : X \to B$ leads to a map $T(f) : T(f^*(\xi)) \to T(\xi)$ called the Thomification of f.

Remark: It can be shown that the homeomorphism type of the Thom space is independent of the choice of metric.

Theorem 6.5.2 *Let $BU(n)$ be the classifying space for the unitary group that is $\pi_k(BU(n)) = \pi_{k-1}U(n)$. There is a unique n-dimensional complex vector bundle $\gamma_n^{\mathbb{C}}$ over it which is universal in the sense that any n-dimensional complex vector bundle over a paracompact space X is induced by a map $X \to BU(n)$ and two such bundles over X are isomorphic if and only if they are induced by homotopic maps. We call $\gamma_n^{\mathbb{C}}$ the universal (n-dimensional complex vector) bundle.*

(See for example [25], pp. 202,203.)

That is we have the following one-to-one correspondence:

$$\begin{array}{c}\{\text{homotopy classes of maps } f : X \to BU(n)\} \\ \updownarrow \\ \{n\text{-dim. complex vector bundles over } X\}\end{array}$$

It can be shown, that $BU(n)$ is the union of the Grassmanian $G_{n,k}^{\mathbb{C}}$ under inclusions maps $i : G_{n,k}^{\mathbb{C}} \to G_{n,k+1}^{\mathbb{C}}$ which are induced by the standard inclusion of $\mathbb{C}^{n+k} \to \mathbb{C}^{n+k+1}$ and send an n-dimensional subspace of $\mathbb{C}^{n+k}$ to the corresponding one in $\mathbb{C}^{n+k+1}$.

On $G^{\mathbb{C}}_{n,k}$ we define a map $j : G^{\mathbb{C}}_{n,k} \to G^{\mathbb{C}}_{n+1,k}$ which is induced by the standard inclusion of $\mathbb{C}^{n+k}$ into $\mathbb{C}^{n+k+1}$ and sends an n-dimensional subspace x of $\mathbb{C}^{n+k}$ to the $(n+1)$-dimensional subspace of $\mathbb{C}^{n+k+1}$ spanned by x and a fixed vector not lying in $\mathbb{C}^{n+k}$. Then $j^*(\gamma^{\mathbb{C}}_{n+1,k}) = \gamma^{\mathbb{C}}_{n,k} \oplus \varepsilon$, where ε denotes the trivial complex line bundle.

Definition 6.5.3 *Let $BU(n)$ be the the classifiying space for the unitary group $U(n)$ and $\gamma^{\mathbb{C}}_n$ the universal bundle over it.*
MU, the Thom spectrum for the unitary group, is defined by

$$MU_{2n} = T(\gamma^{\mathbb{C}}_n) \text{ and } MU_{2n+1} = \Sigma MU_{2n}.$$

The map $\Sigma MU_{2n} \to MU_{2n+1}$ is the obvious one. In order to get a map

$$\Sigma^2 T(\gamma^{\mathbb{C}}_n) = \Sigma MU_{2n+1} \to MU_{2n+2} = T(\gamma^{\mathbb{C}}_{n+1}).$$

we consider the map $j : BU(n) \to BU(n+1)$ with $j^*(\gamma^{\mathbb{C}}_{n+1}) = \gamma_n \oplus \varepsilon$. The Thomification of j is by corollary (6.4.6) the desired map:

$$T(j) : T(j^*(\gamma^{\mathbb{C}}_{n+1})) = \Sigma^2 T(\gamma^{\mathbb{C}}_n) \to T(\gamma^{\mathbb{C}}_{n+1})$$

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