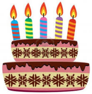
Suslin Proper Forcing and Regularity Properties.

Yurii Khomskii
KGRC, Vienna

based on joint work with Vera Fischer and Sy Friedman



Colloquium Logicum
München, 5 September 2014

Purpose of this talk

Some time ago, in joint work with Vera Fischer and Sy Friedman, we made substantial progress in separating various regularity properties on the $\boldsymbol{\Delta}^1_3$-level of the projective hierarchy. The crucial thing was obtaining models where all $\boldsymbol{\Delta}^1_3$-sets satisfied a certain regularity property, but in a "minimal" way. Today I will just talk about **one of these methods**, *focusing on* **one particular regularity property** *as a canonical example.*

Bernstein property

Question

Suppose A is a subset of 2^ω. Is there a perfect set P such that $P \subseteq A$ or $P \cap A = \varnothing$?

(Perfect set = homeomorphic copy of 2^ω; equivalently $P = [T]$ for a perfect tree $T \subseteq 2^{<\omega}$).

Bernstein property

Some answers:

- If A is **open** or **closed**, yes (trivial).

Bernstein property

Some answers:

- If A is **open** or **closed**, yes (trivial).
- If A is **analytic**, yes. (Suslin 1917).

Bernstein property

Some answers:

- If A is **open** or **closed**, yes (trivial).
- If A is **analytic**, yes. (Suslin 1917).
- For **arbitrary** A, no (Bernstein 1908).

  Proof: Enumerate all perfect sets $\{P_\alpha \mid \alpha < 2^{\aleph_0}\}$ and construct A by "diagonalization".

Abstract Bernstein property

Definition

Suppose $\mathbb{P}$ is a collection of subsets of ω^ω or 2^ω. A set A is called $\mathbb{P}$-**measurable** iff there exists $P \in \mathbb{P}$ such that $P \subseteq A$ or $P \cap A = \varnothing$.

Abstract Bernstein property

Definition

Suppose $\mathbb{P}$ is a collection of subsets of ω^ω or 2^ω. A set A is called $\mathbb{P}$-**measurable** iff there exists $P \in \mathbb{P}$ such that $P \subseteq A$ or $P \cap A = \varnothing$.

Interesting examples are when $\mathbb{P}$ = forcing poset on the reals.

Abstract Bernstein property

Definition

Suppose $\mathbb{P}$ is a collection of subsets of ω^ω or 2^ω. A set A is called $\mathbb{P}$-**measurable** iff there exists $P \in \mathbb{P}$ such that $P \subseteq A$ or $P \cap A = \varnothing$.

Interesting examples are when $\mathbb{P}$ = forcing poset on the reals.

Definition

Sacks forcing $\mathbb{S}$ is the partial order of perfect trees on $2^{<\omega}$ ordered by inclusion.

So Bernstein property = "$\mathbb{S}$-measurability".

Why?

Why is this interesting?

Why?

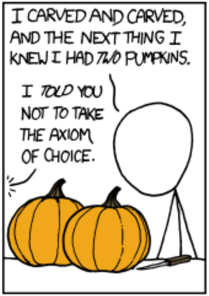
Why is this interesting?

It provides a framework for many **regularity properties** for sets of reals, e.g., **Lebesgue measurability**, **property of Baire**, **Ramsey property**, etc.

Banach-Tarski paradox

Most of you have probably heard this "paradox":

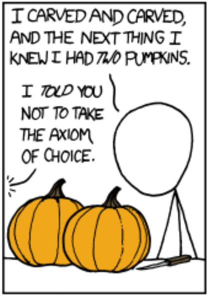
You can take a sphere, cut it up into five pieces, rearrange the pieces using only the operations of rotation and translation (no stretching), and assemble them back to form two spheres of the same size as the original.



Banach-Tarski paradox

Most of you have probably heard this "paradox":

> *You can take a sphere, cut it up into five pieces, rearrange the pieces using only the operations of rotation and translation (no stretching), and assemble them back to form two spheres of the same size as the original.*



Lebesgue-measurability can be described as $\mathbb{P}$-measurability, for $\mathbb{P} =$ random forcing.

A modern proof of Suslin's theorem

Theorem (Suslin 1917)

All analytic sets have the Bernstein Property.

Modern proof.

Let $A = \{x \mid \phi(x)\}$, with ϕ a Σ^1_1 formula. Let $\dot{x}_G$ be the name for the Sacks-generic real, and let T be a Sacks-condition deciding $\phi(\dot{x}_G)$, w.l.o.g. $T \Vdash \phi(\dot{x}_G)$. Let $M \prec \mathcal{H}_\theta$ be a **countable elementary submodel** with $\mathbb{S}, T \in M$. Using a **properness argument** find $S \leq T$ such that all $x \in [S]$ are Sacks-generic over M. So for all $x \in [S]$, $M[x] \models \phi(x)$, and by $\boldsymbol{\Sigma}^1_1$**-absoluteness** $\phi(x)$. Therefore $[T] \subseteq A$. $\square$

Beyond analytic

Theorem (Gödel 1938)

In L, there is a $\mathbf{\Delta}^1_2$-set without the Bernstein property.

Proof.

Again diagonalize against all perfect trees, but use the $\mathbf{\Sigma}^1_2$-**good wellorder of the reals** of L. □

Beyond analytic

Theorem (Folklore)

After an ω_1-itersation of $\mathbb{S}$, all $\mathbf{\Delta}^1_2$ sets have the Bernstein Property.

Proof.

Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$, w.l.o.g. parameters in V. The statement $\forall x\ (\phi(x) \leftrightarrow \neg\psi(x))$ is $\mathbf{\Pi}^1_3$ hence downward absolute between $V^{\mathbb{S}_{\omega_1}}$ and $V^{\mathbb{S}}$. In V find Sacks-condition T forcing $\phi(\dot{x}_G)$ or $\psi(\dot{x}_G)$, and proceed as before (and use **upwards $\mathbf{\Sigma}^1_2$-absoluteness** from $M[x]$ to $V^{\mathbb{S}_{\omega_1}}$). □

Beyond analytic

Theorem (Folklore)

After an ω_1-itersation of $\mathbb{S}$, all $\boldsymbol{\Delta}^1_2$ sets have the Bernstein Property.

Proof.

Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$, w.l.o.g. parameters in V. The statement $\forall x\, (\phi(x) \leftrightarrow \neg\psi(x))$ is $\boldsymbol{\Pi}^1_3$ hence downward absolute between $V^{\mathbb{S}_{\omega_1}}$ and $V^{\mathbb{S}}$. In V find Sacks-condition T forcing $\phi(\dot{x}_G)$ or $\psi(\dot{x}_G)$, and proceed as before (and use **upwards $\boldsymbol{\Sigma}^1_2$-absoluteness** from $M[x]$ to $V^{\mathbb{S}_{\omega_1}}$). $\square$

Remark: It is not hard to do better and obtain $V^{\mathbb{S}_{\omega_1}} \models \boldsymbol{\Sigma}^1_2(\mathbb{S})$.

What about $\mathbf{\Delta}^1_3$?

Question

Can we use similar methods to obtain the result for $\mathbf{\Delta}^1_3$ and higher levels?

Problems:

1. We used **Shoenfield absoluteness** and $\mathbf{\Sigma}^1_1$**-absoluteness** for countable models.
2. Using coding techniques (e.g. "almost disjoint coding") one can force a "Σ^1_3-good wellorder of the reals" over L, obtaining a model where not all $\mathbf{\Delta}^1_3$ sets have the Bernstein property.

This suggests that the **definability** of the forcing iteration plays a role.

Properness Without Elementaricity

Definition (Judah, Shelah, Goldstern)

A forcing $(\mathbb{P}, \leq)$ is **Suslin proper** if

1. elements of $\mathbb{P}$ are (coded by) reals,
2. "$p \in \mathbb{P}$", "$p \leq q$" and "$p \perp q$" are $\mathbf{\Sigma}^1_1$ relations.
3. a strong version of properness holds, where "for all $M \prec \mathcal{H}_\theta$" is replaced by "for all countable models M of a sufficient fragment of ZFC".

Properness Without Elementaricity

Definition (Judah, Shelah, Goldstern)

A forcing $(\mathbb{P}, \leq)$ is **Suslin proper** if

1. elements of $\mathbb{P}$ are (coded by) reals,
2. "$p \in \mathbb{P}$", "$p \leq q$" and "$p \perp q$" are $\mathbf{\Sigma}^1_1$ relations.
3. a strong version of properness holds, where "for all $M \prec \mathcal{H}_\theta$" is replaced by "for all countable models M of a sufficient fragment of ZFC".

"Suslin$^+$ proper" is a slight modification needed for technical reasons.

Properness Without Elementaricity

Definition (Judah, Shelah, Goldstern)

A forcing $(\mathbb{P}, \leq)$ is **Suslin proper** if

1. elements of $\mathbb{P}$ are (coded by) reals,
2. "$p \in \mathbb{P}$", "$p \leq q$" and "$p \perp q$" are $\mathbf{\Sigma}^1_1$ relations.
3. a strong version of properness holds, where "for all $M \prec \mathcal{H}_\theta$" is replaced by "for all countable models M of a sufficient fragment of ZFC".

"Suslin$^+$ proper" is a slight modification needed for technical reasons.

All standard definable forcings used in the theory of the reals which are known to be proper, are actually Suslin$^+$ proper.

Jakob Kellner, Preserving non-null with Suslin$^+$ forcings, Arch. Math. Logic (2006) 45:649–664.

Properness Without Elementaricity

Lemma (Fischer-Friedman-Kh or Folklore?)

Let $\mathbb{P}$ be Suslin$^+$ proper and τ a nice $\mathbb{P}$-name for a real. Then for any $\mathbf{\Pi}^1_n$-formula θ, the statement "$p \Vdash_{\mathbb{P}} \theta(\tau)$" is also $\mathbf{\Pi}^1_n$, for all $n \geq 2$.

The same applies for **iterations** of Suslin$^+$ proper forcing notions of length ω_1.

Main Result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Main Result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof-sketch.

Main Result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof-sketch.

Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be $\mathbf{\Delta}^1_3$, w.l.o.g. parameters in V.

Let x_0 be first Sacks-generic over V. W.l.o.g. $V[G_{\omega_1}] \models \phi(x_0)$. Then $V[G_{\omega_1}] \models \exists y \theta(x_0, y)$ for some $\mathbf{\Pi}^1_2$ formula θ. By properness, there is $\alpha < \omega_1$ such that $y \in V[G_\alpha]$, and by Shoenfield absoluteness $V[G_\alpha] \models \theta(x_0, y)$. In V, let p be a $\mathbb{S}_\alpha$-condition and τ a nice $\mathbb{S}_\alpha$-name for a real, such that

$$p \Vdash_\alpha \theta(\dot{x}_{G(0)}, \tau).$$

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

Then $V[x_0] \models$ "if we force with the remainder $\mathbb{S}_{1,\alpha} \cong \mathbb{S}_\alpha$ along p then $\theta(\check{x}_0, \tau[x_0])$ will hold".

Let $\tilde{\theta}(x)$ abbreviate the above statement. Using properties of "Suslin$^+$ proper forcings", it turns out that $\tilde{\theta}$ is $\mathbf{\Pi}^1_2$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t. $\forall x \in [T]\,(x \text{ is } \mathbb{S}\text{-generic over } V)$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t.
$\forall x \in [T]\,(x \text{ is } \mathbb{S}\text{-generic over } V)$.
$\forall x \in [T]\,(V[x] \models \tilde{\theta}(x))$,

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t.
$\forall x \in [T]$ (x is $\mathbb{S}$-generic over V).
$\forall x \in [T]$ $(V[x] \models \tilde{\theta}(x))$,
$\forall x \in [T]$ $\tilde{\theta}(x)$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t.
$\forall x \in [T]$ (x is $\mathbb{S}$-generic over V).
$\forall x \in [T]$ $(V[x] \models \tilde{\theta}(x))$,
$\forall x \in [T]\, \tilde{\theta}(x)$.

Let $\Theta(T)$ abbreviate "$\forall x \in [T]\, \tilde{\theta}(x)$". This is $\mathbf{\Pi}^1_2$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t.
$\forall x \in [T]\,(x$ is $\mathbb{S}$-generic over $V)$.
$\forall x \in [T]\,(V[x] \models \tilde{\theta}(x))$,
$\forall x \in [T]\,\tilde{\theta}(x)$.

Let $\Theta(T)$ abbreviate "$\forall x \in [T]\,\tilde{\theta}(x)$". This is $\mathbf{\Pi}^1_2$.

Then $V[G_\beta] \models \Theta(T)$ for $1 \leq \beta < \omega_1$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

We claim $V[G_{\omega_1}] \models [T] \subseteq A$.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

We claim $V[G_{\omega_1}] \models [T] \subseteq A$.

Pick $z \in [T]$, let $\beta < \omega_1$ be such that $z \in V[G_\beta]$. Since $V[G_\beta] \models \Theta(T)$ in particular $V[G_\beta] \models \tilde{\theta}(z)$, so in particular

$$V[G_\beta] \models p[z] \Vdash_{\mathbb{S}_\alpha} \theta(\check{z}, \tau[z]).$$

By genericity we may assume β is sufficiently large so that $p[z]$ is in the generic.

Main result

Theorem (Fischer-Friedman-Kh)

After an ω_1-iteration of Sacks forcing, all $\mathbf{\Delta}^1_3$-sets have the Bernstein property.

Proof.

We claim $V[G_{\omega_1}] \models [T] \subseteq A$.

Pick $z \in [T]$, let $\beta < \omega_1$ be such that $z \in V[G_\beta]$. Since $V[G_\beta] \models \Theta(T)$ in particular $V[G_\beta] \models \tilde{\theta}(z)$, so in particular

$$V[G_\beta] \models p[z] \Vdash_{\mathbb{S}_\alpha} \theta(\check{z}, \tau[z]).$$

By genericity we may assume β is sufficiently large so that $p[z]$ is in the generic.

It follows that $V[G_{\beta+\alpha}] \models \theta(z, \tau[z][G_{[\beta+1,\beta+\alpha)}])$, hence $V[G_{\beta+\alpha}] \models \phi(z)$, and by upwards-absoluteness, $V[G_{\omega_1}] \models \phi(z)$. $\square$

Possible generalizations

Have we used anything specific about Sacks forcing?

Possible generalizations

Have we used anything specific about Sacks forcing?

Only: if you add a Sacks-real you add a perfect set of Sacks-reals.

Amoeba and Quasi-amoeba

Let $\mathbb{P}$ be a forcing whose conditions are trees on 2^ω or ω^ω ordered by inclusion. Let $\mathbb{AP}$ be some other forcing.

Definition

1. We say that $\mathbb{AP}$ is a **quasi-amoeba for** $\mathbb{P}$ if $\mathbb{AP}$ adds a $\mathbb{P}$-tree of $\mathbb{P}$-generic reals.
2. We say that $\mathbb{AP}$ is an **amoeba for** $\mathbb{P}$ if $\mathbb{AP}$ adds a $\mathbb{P}$-tree of $\mathbb{P}$-generic reals **and this remains true in further extensions**.

Amoeba and Quasi-amoeba

For Cohen and random, **quasi-amoeba** and **amoeba** are the same thing, but in general they are different.

Amoeba and Quasi-amoeba

For Cohen and random, **quasi-amoeba** and **amoeba** are the same thing, but in general they are different.

Examples:

1. Sacks forcing is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
2. Miller forcing is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
3. Mathias forcing is an amoeba for itself.

General theorem

Theorem (Fischer-Friedman-Kh)

Suppose $\mathbb{P}$ is a tree-like forcing, $\mathbb{AP}$ a **quasi-amoeba** *for $\mathbb{P}$, and both $\mathbb{P}$ and $\mathbb{AP}$ are* **Suslin$^+$ proper**. *Then after iterating with $(\mathbb{P} * \mathbb{AP})_{\omega_1}$, all $\mathbf{\Delta}^1_3$-sets are $\mathbb{P}$-measurable.*

Application

$$\begin{tikzcd}
\boldsymbol{\Delta}^1_3(\mathbb{B}) \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_3(\mathbb{V}) \arrow[rr, Rightarrow] & & \boldsymbol{\Delta}^1_3(\mathbb{S}) \\
\boldsymbol{\Delta}^1_3(\mathbb{R}) \arrow[ur, Rightarrow] \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
& & \boldsymbol{\Delta}^1_3(\mathbb{C}) \arrow[uul, Rightarrow] \arrow[u, Rightarrow] &
\end{tikzcd}$$

$\mathbb{C}$ = Baire property; $\mathbb{B}$ = Lebesgue measure; $\mathbb{S}$ = Sacks-measurability; $\mathbb{M}$ = Miller-measurability; $\mathbb{L}$ = Laver-measurability; $\mathbb{V}$ = Silver measurability; $\mathbb{R}$ = Ramsey property.

Theorem (Fischer-Friedman-Kh)

Each constellation of "true"/"false" assignments (18 possibilities) to the above statements not contradicting this diagram, is consistent r.t. ZFC or ZFC + inaccessible.

Thank you!

Yurii Khomskii
yurii@deds.nl