# Definable Hausdorff Gaps

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# Definitions

Notation:

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- $=^*$: equality modulo finite
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- A pair $(A, B)$ is a **gap** if it is a pre-gap which cannot be separated.

# Types of gaps

**Theorem (Hausdorff 1936)**

*There exists an* $\boldsymbol{(\omega_1, \omega_1)}$**-gap** $(A, B)$*: $A$ and $B$ well-ordered by* $\subseteq^*$*, with order-type* $\omega_1$*.*

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**Theorem (Todorčević 1996)**

*There exists a* **perfect gap** $(A, B)$*: both A and B are perfect sets.*

**Proof.**

$A := \{\{x \restriction n \mid x(n) = 0\} \mid x \in 2^\omega\} \subseteq [\omega^{<\omega}]^\omega$
$B := \{\{x \restriction n \mid x(n) = 1\} \mid x \in 2^\omega\} \subseteq [\omega^{<\omega}]^\omega$. □

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We will say that a gap $(A, B)$ is a **Hausdorff gap** if $A$ and $B$ are $\sigma$-directed (every countable subset has an $\subseteq^*$-upper bound).

**Theorem (Todorčević 1996)**

*If either $A$ or $B$ is analytic then $(A, B)$ cannot be a Hausdorff gap.*

# Proof

About the proof:

- $A$ and $B$ are **$\sigma$-separated** if $\exists C$ countable s.t. $C \perp B$ and $\forall a \in A \, \exists c \in C \, (a \subseteq^* c)$

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- $A$ and $B$ are **$\sigma$-separated** if $\exists C$ countable s.t. $C \perp B$ and $\forall a \in A \, \exists c \in C \, (a \subseteq^* c)$
- A tree $S$ on $\omega^{\uparrow\omega}$ is an **$(\mathbf{A}, \mathbf{B})$-tree** if
  1. $\forall \sigma \in S: \; \{i \mid \sigma^\frown \langle i \rangle \in S\}$ has infinite intersection with some $b \in B$,
  2. $\forall x \in [S]: \; \mathrm{ran}(x) \subseteq^* a$ for some $a \in A$.

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**Point:**

1. If $A$ is $\sigma$-directed, then "$\sigma$-separated" $\rightarrow$ "separated".
2. If $B$ is $\sigma$-directed, then there is no $(A, B)$-tree.

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*If $A$ is analytic then either there exists an $(A, B)$-tree or $A$ and $B$ are $\sigma$-separated.*

# Extending this result

We can extend this in various directions.

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2. Determinacy
3. $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$ level

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Probably there are other proofs...

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$$\begin{array}{c|cccccccc} \mathrm{I}: & c_0 & & (s_1, c_1) & & (s_2, c_2) & & \dots & \\ \hline \mathrm{II}: & & i_0 & & i_1 & & i_2 & & \dots \end{array}$$

where $s_n \in \omega^{<\omega}$, $c_n \in [\omega]^\omega$ and $i_n \in \omega$. The conditions for player I:

1. $\min(s_n) > \max(s_{n-1})$ for all $n \geq 1$,
2. $\min(c_n) > \max(s_n)$,
3. all $c_n$ have infinite intersection with some $b \in B$, and
4. $i_n \in \mathrm{ran}(s_{n+1})$ for all $n$.

Conditions for player II:

1. $i_n \in c_n$ for all $n$.

If all five conditions are satisfied, let $s^* := s_1{}^\frown s_2{}^\frown \dots$. Player I wins iff $\mathrm{ran}(s^*) \in A$.

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Unfortunately, I don't know how to do it with AD!

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**Theorem (Kh)**

*The following are equivalent:*

1. *there is no $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap*
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5. $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$

Non-trivial directions: (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

# Proof

**(5)** $\Rightarrow$ **(1)** : $\forall r\,(\aleph_1^{L[r]} < \aleph_1) \;\Rightarrow\; \nexists(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap.

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**Lemma (Kh)**

*If $A$ is $\Sigma^1_2(r)$ then either there exists an $(A, B)$-tree or $A$ and $B$ are $C$-separated by some $C \subseteq L[r]$.*

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Hence: if $\omega^\omega \cap L[r]$ is countable then $C$ is countable, so "$C$-separated" $\Rightarrow$ "$\sigma$-separated".

# Proof (continued)

**(4)** $\Rightarrow$ **(5)** : $\exists r\,(\aleph_1^{L[r]} = \aleph_1) \;\Rightarrow\; \exists(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap.

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**Point:** A gap satisfying HC is **indestructible**, i.e., remains a gap in any larger model $W \supseteq V$ as long as $\aleph_1^W = \aleph_1^V$.

# Proof (continued)

**Lemma (Hausdorff)**: if initial segment $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ satisfies HC, then we can find $a_\alpha, b_\alpha$ so that $(\{a_\gamma \mid \gamma \leq \alpha\}, \{b_\gamma \mid \gamma \leq \alpha\})$ still satisfies HC.

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Do this in any $L[r]$, get $\boldsymbol{\Sigma}^1_2$ definitions for $A$ and $B$ (choose $<_{L[r]}$-least $a_\alpha, b_\alpha$).

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Do this in any $L[r]$, get $\mathbf{\Sigma}^1_2$ definitions for $A$ and $B$ (choose $<_{L[r]}$-least $a_\alpha, b_\alpha$).

Assuming $\aleph_1^{L[r]} = \aleph_1$, we get a $(\Sigma^1_2(r), \Sigma^1_2(r))$-Hausdorff gap (in $V$).

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For more about this, please wait $\pm 10$ min!

# Coding Lemma

**Coding Lemma (Kh)**

If an initial segment $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ satisfies HC, then we can find $a_\alpha, b_\alpha$ so that $(\{a_\gamma \mid \gamma \leq \alpha\}, \{b_\gamma \mid \gamma \leq \alpha\})$ still satisfies HC, and **additionally** both $a_\alpha$ and $b_\alpha$ recursively code an arbitrary countable model $M$.

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Do this in $L[r]$ with $\aleph_1^{L[r]} = \aleph_1$, and obtain a $(\Pi^1_1(r), \Pi^1_1(r))$-Hausdorff gap (in $V$). □

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2. Can we get rid of Miller's method (purely methodological interest).
3. Higher projective levels (e.g. $\mathbf{\Sigma}^1_{n+1}$ vs. $\mathbf{\Pi}^1_n$)?

# Dziękuję za uwagę!

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- Felix Hausdorff, *Summen von $\aleph_1$ Mengen*, Fundamenta Mathematicae 26 (1936), pp. 241–255.
- Arnold Miller, *Infinite combinatorics and definability*, Annals of Pure and Applied Logic 41 (1989), pp. 179–203.
- Stevo Todorčević, *Analytic gaps*, Fundamenta Mathematicae 150, No. 1 (1996), pp. 55–66.