Polarized Partition Properties for $\mathbf{\Delta}^1_2$ sets.

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"*More regularity on $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$-level* $\quad\propto\quad$ **L** *gets smaller*"

Examples

1. $\mathbf{\Delta}^1_2(\text{Lebesgue}) \iff \forall a\ \exists$ random-generic$/\mathbf{L}[a]$
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3. $\mathbf{\Delta}^1_2$(Ramsey) $\iff \forall a \, \exists$ Ramsey real$/\mathbf{L}[a]$
4. $\mathbf{\Delta}^1_2$(Laver) $\iff \forall a \, \exists$ dominating real$/\mathbf{L}[a]$
5. $\mathbf{\Delta}^1_2$(Miller) $\iff \forall a \, \exists$ unbounded real$/\mathbf{L}[a]$
6. $\mathbf{\Delta}^1_2$(Sacks) $\iff \forall a \, \exists$ real $\notin \mathbf{L}[a]$

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Given two regularity properties: Reg_1 and Reg_2, we are interested in:

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Brendle, Löwe, Ikegami: Regularity based on forcing.

We consider Ramsey-theoretic partition properties (on the reals).

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- For $x, y \in \omega^\omega$, a set $A \subseteq \omega^\omega$ satisfies $((x) \to (y))$ (*bounded polarized partition*) if

$$\exists H, \|H\| = y \text{ and } \forall i\ H(i) \subseteq x(i) \text{ s.t. } [H] \subseteq A \vee [H] \cap A = \varnothing$$

  Here, x is explicitly definable from y (the $x(n)$'s are recursive in the $y(n)$'s).

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Our question: What about $\mathbf{\Delta}^1_2((\bar{\omega}) \longrightarrow (y))$ and $\mathbf{\Delta}^1_2((x) \longrightarrow (y))$?

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3. For all $y, y' (\geq 2)$: if $\mathbf{\Delta}^1_2((x) \to (y))$ for some x, then $\mathbf{\Delta}^1_2((x') \to (y'))$ for some x'.

   Use coding function $\varphi(x) := \langle \langle x(0), \ldots, x(i_1) \rangle, \langle x(i_1+1), \ldots, x(i_1+i_2) \rangle, \ldots \rangle$.

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   Use coding function $\varphi(x) := \langle \langle x(0), \ldots, x(i_1) \rangle, \langle x(i_1+1), \ldots, x(i_1+i_2) \rangle, \ldots \rangle$.
4. $\mathbf{\Delta}^1_2(\text{Ramsey}) \implies \mathbf{\Delta}^1_2((\bar{\omega}) \to (y))$.

   Given A, let $X \in \omega^{\uparrow\omega}$ be homogeneous for $A \cap \omega^{\uparrow\omega}$. Then divide ran$(X)$ into $X_0, X_1, \ldots$ such that $|X_n| = y(n)$. Now $H := \langle X_0, X_1, \ldots \rangle$ witnesses that A is $((\bar{\omega}) \to (y))$-measurable.

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Theorem. If $\mathbf{\Delta}^1_2((\bar{\omega}) \to (y))$ then $\forall a$ there is an eventually different real over **L**$[a]$.

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- W.l.o.g., assume that $\forall x \; \exists y \in \mathbf{L}[a]$ s.t. $\exists^\infty n \; [x(n) = y(n) \;\&\; x(n+1) = y(n+1)]$. Let y_x denote the $<_{\mathbf{L}[a]}$-least such real.

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- Let $A := \{x \mid$ first n at which $x(n) = y_x(n)$ is even$\}$. This is $\Delta^1_2(a)$ using the fact that $<_{\mathbf{L}[a]}$ is $\Delta^1_2(a)$.

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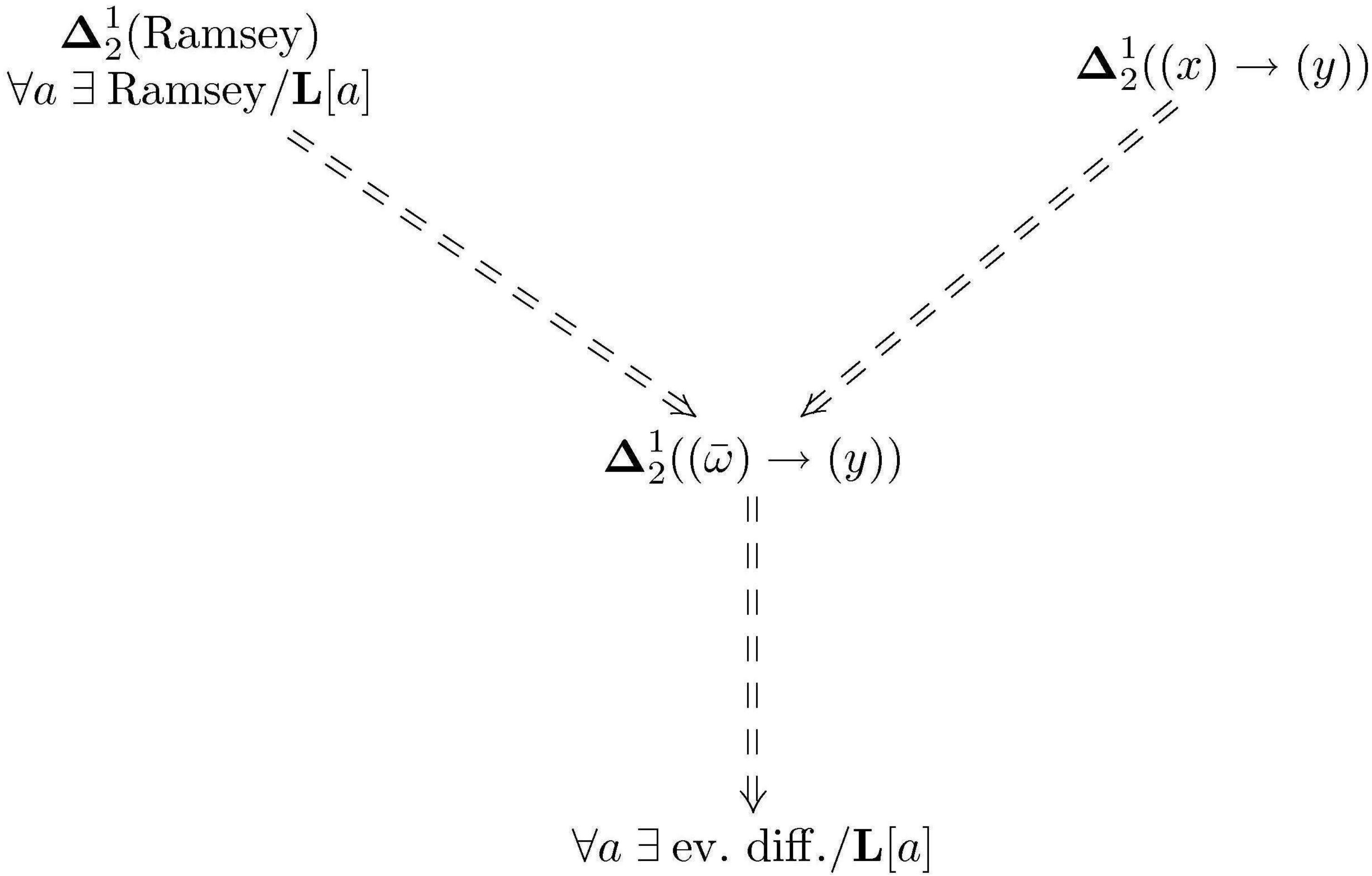
- Let H be homogeneous for A, w.l.o.g. $[H] \subseteq A$. But if $x \in [H]$ then let us change finitely many digits of x to produce a new real x', such that the first n at which $x'(n) = y_x(n)$ is odd but still $x' \in [H]$. It is easy to see that $y_x = y_{x'}$, hence $x' \notin A$: contradiction. □

Diagram of implications



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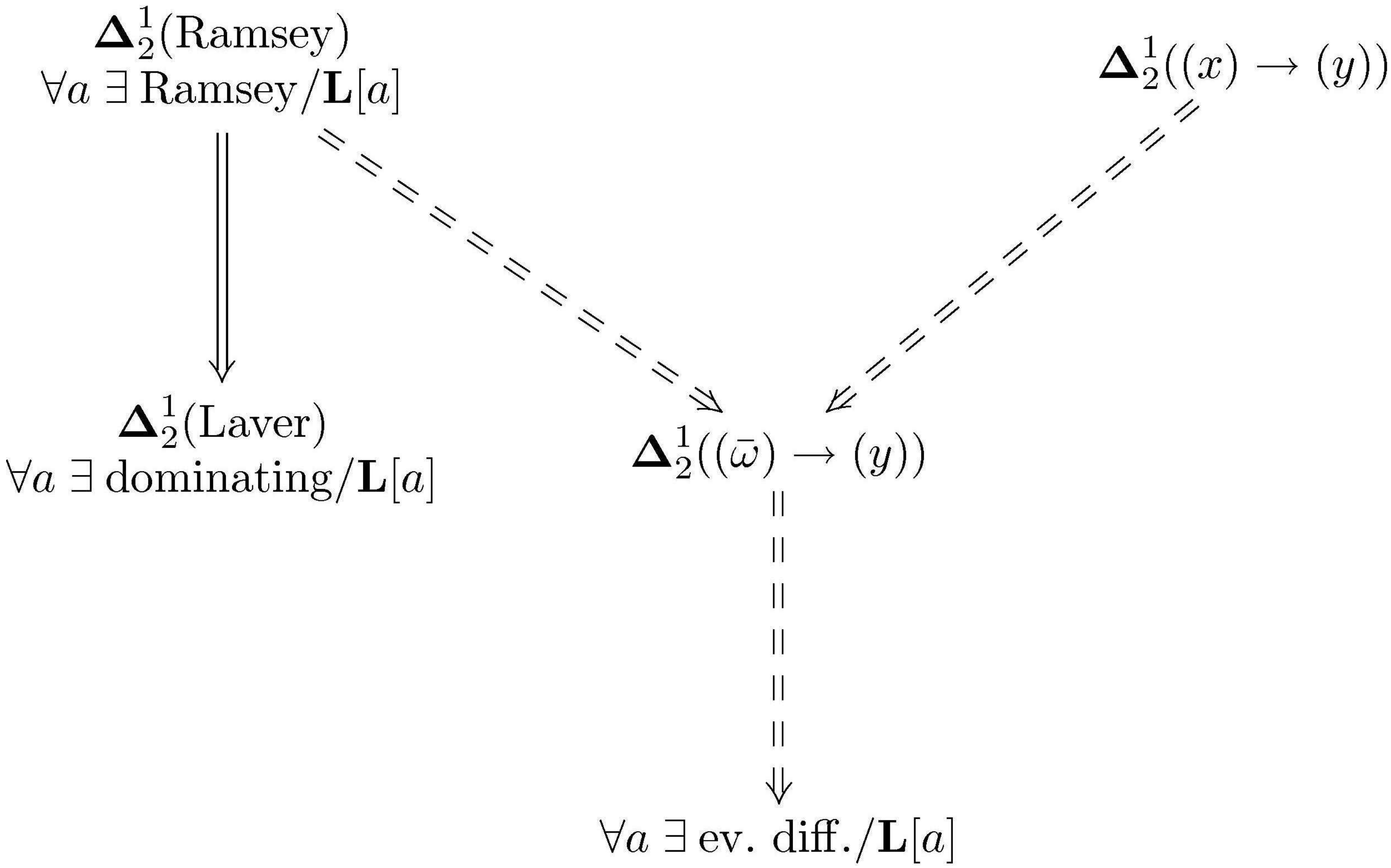

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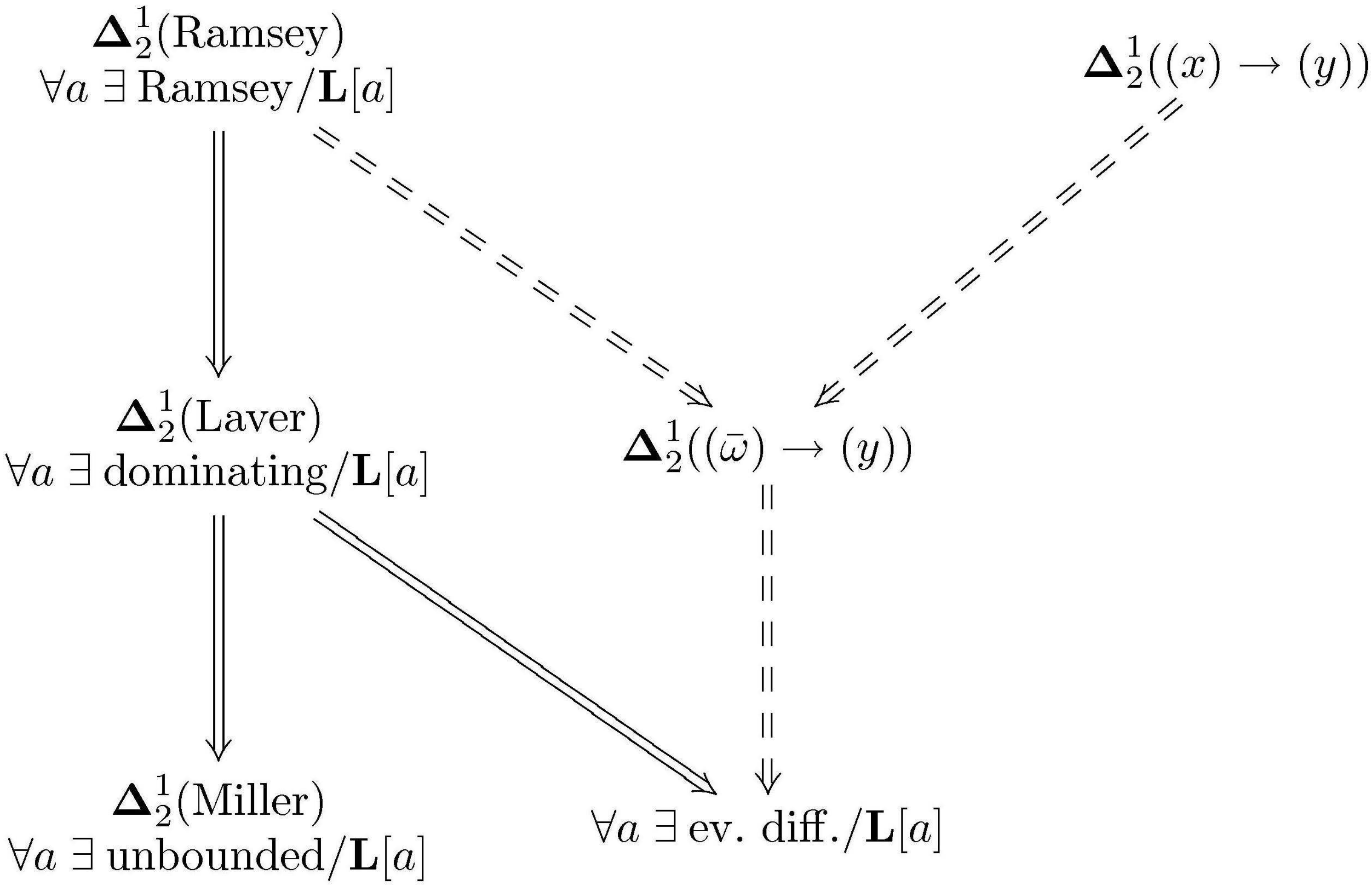

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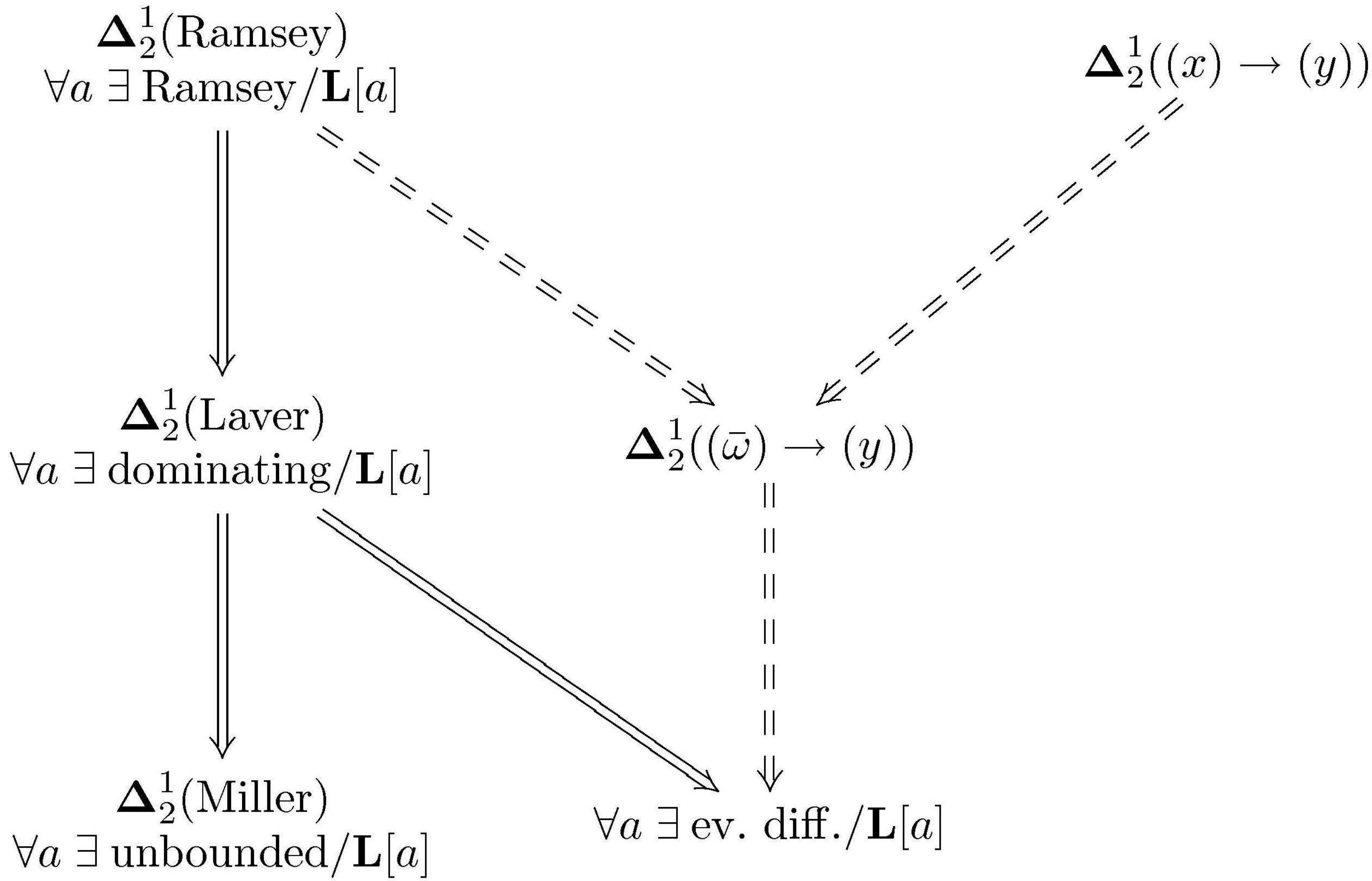

Question: which implications cannot be reversed?

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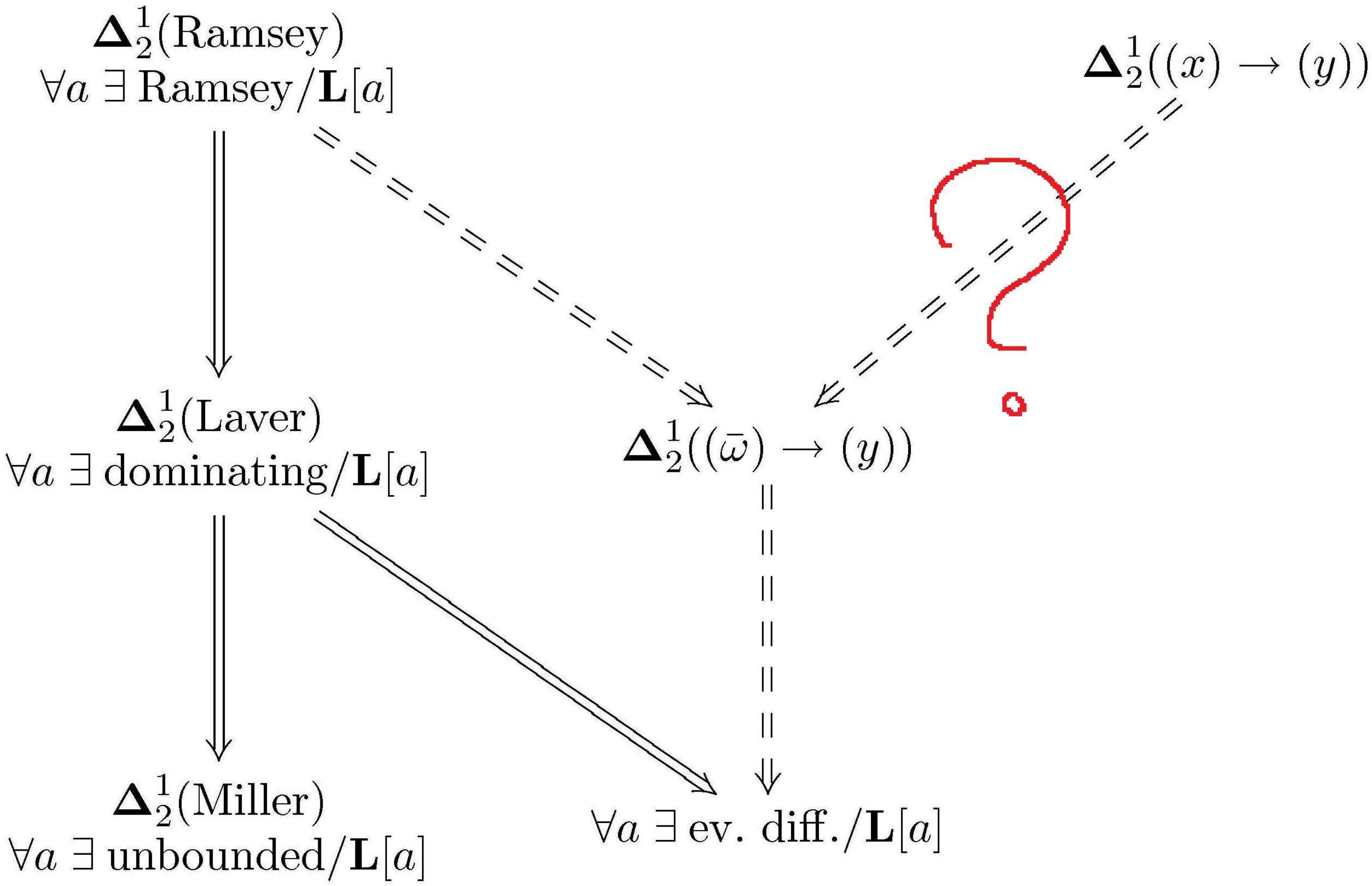

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Theorem. Let $\mathbf{L}^{\mathbb{R}_{\omega_1}}$ be the *Mathias model*, i.e., the ω_1-iteration with countable support of Mathias forcing starting from **L**. Then $\mathbf{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\text{Ramsey})$ but $\neg\mathbf{\Delta}^1_2((x) \to (y))$.

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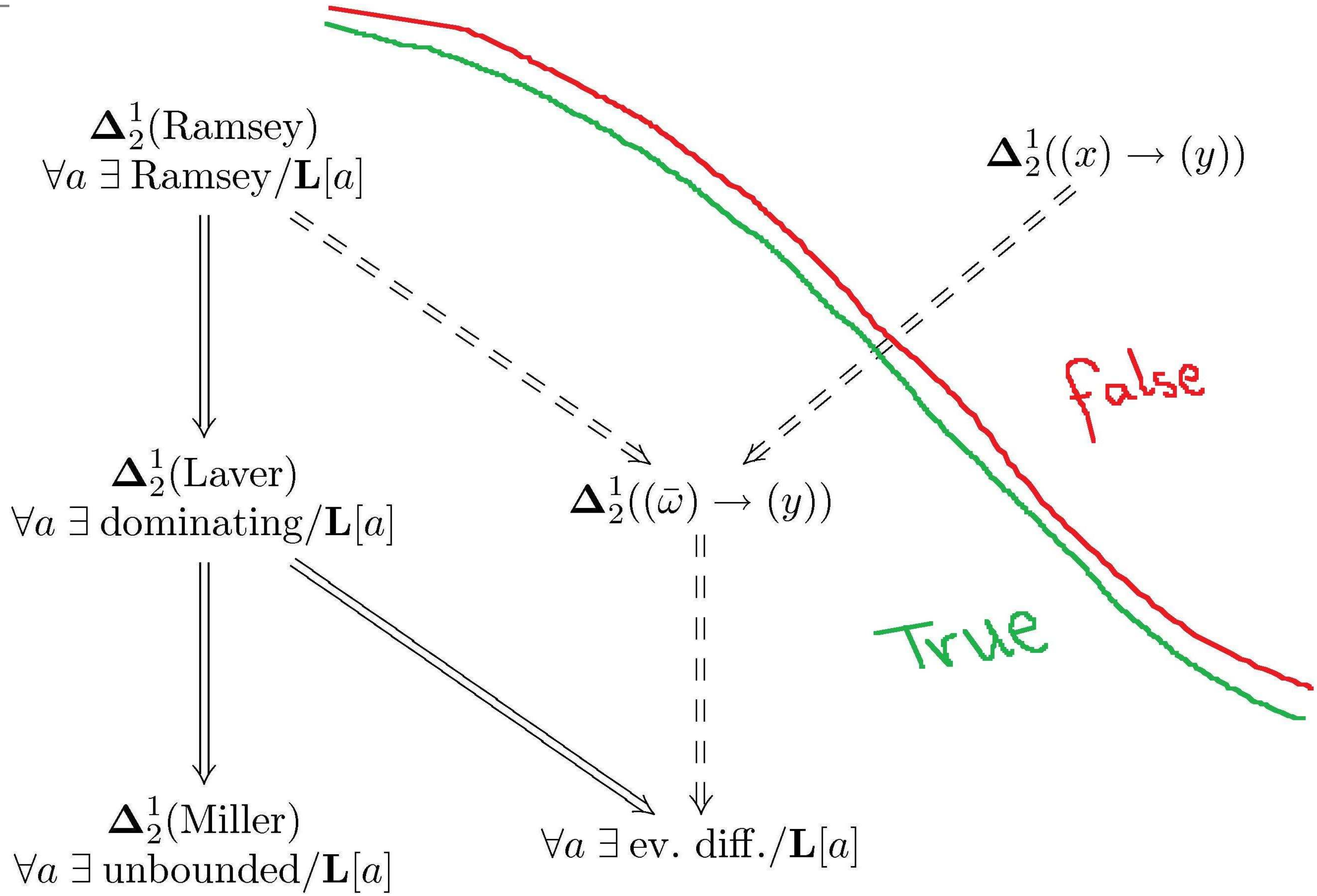

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Proof

- Clearly $\mathbf{\Delta}^1_2$(Ramsey) holds in $\mathbf{L}^{\mathbb{R}_{\omega_1}}$.
- Let $\mathcal{C} := \{S : \omega \longrightarrow [\omega]^{<\omega} \mid \forall n |S(n)| \leq 2^n\}$. Mathias forcing satisfies the *Laver property*: For every $y \in M \cap \omega^\omega$ and $\dot{x}$ s.t. $\Vdash \forall n\ \dot{x}(n) \leq y(n)$, there is an $S \in \mathcal{C} \cap M$ s.t. $\Vdash \forall n\ \dot{x}(n) \in S(n)$.

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- Use the $\mathbf{\Delta}^1_2$ well-ordering of $\mathbf{L} \cap \omega^\omega$ to define a $\mathbf{\Delta}^1_2$ well-ordering of $\mathbf{L} \cap \mathcal{C}$.

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- Use the $\mathbf{\Delta}^1_2$ well-ordering of $\mathbf{L} \cap \omega^\omega$ to define a $\mathbf{\Delta}^1_2$ well-ordering of $\mathbf{L} \cap \mathcal{C}$.
- Use that to define a $\mathbf{\Delta}^1_2$ set A which explicitly violates $((x) \rightarrow (y))$, where y grows faster then 2^n. This set is well-defined because of the Laver property. □

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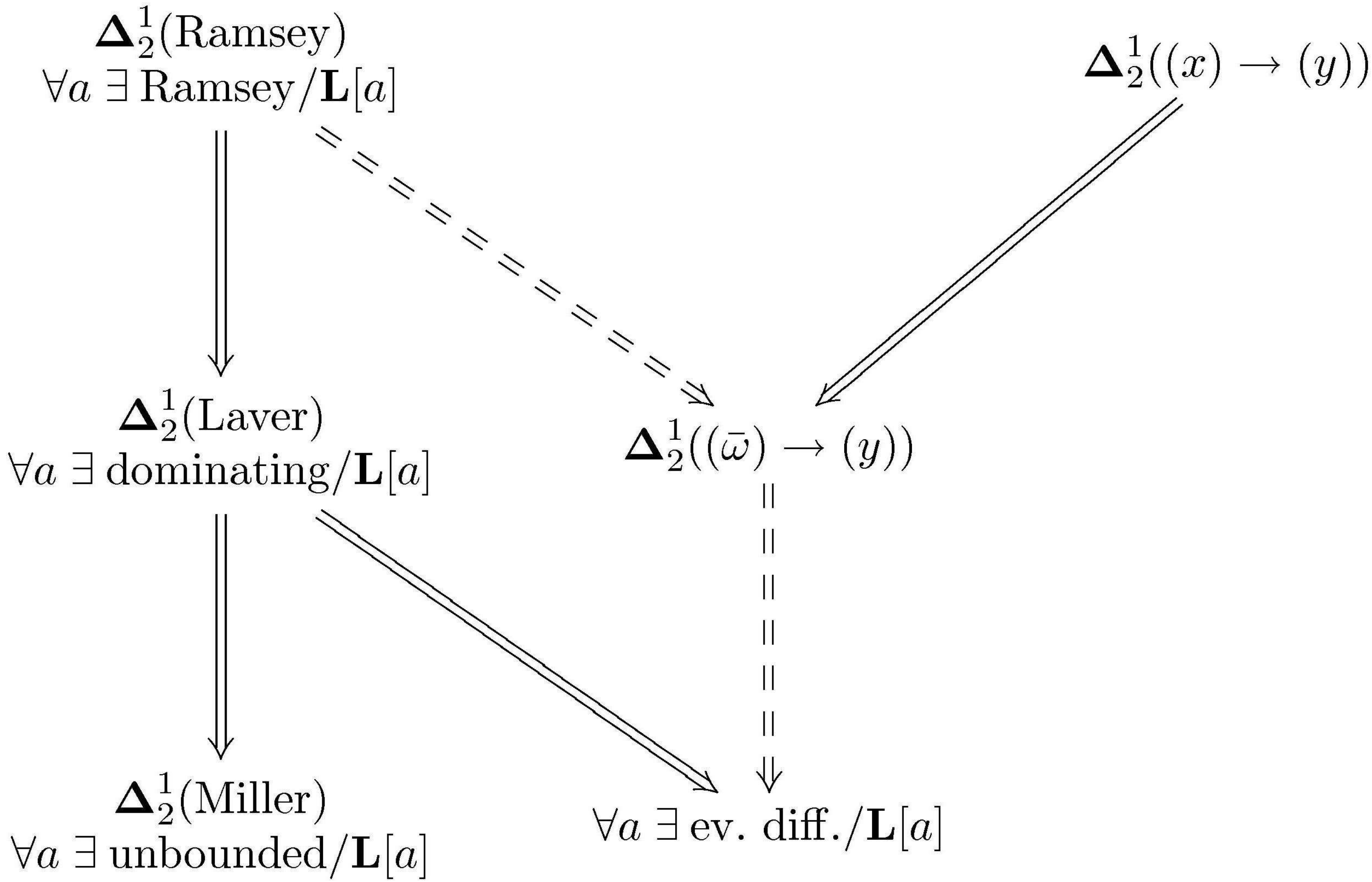

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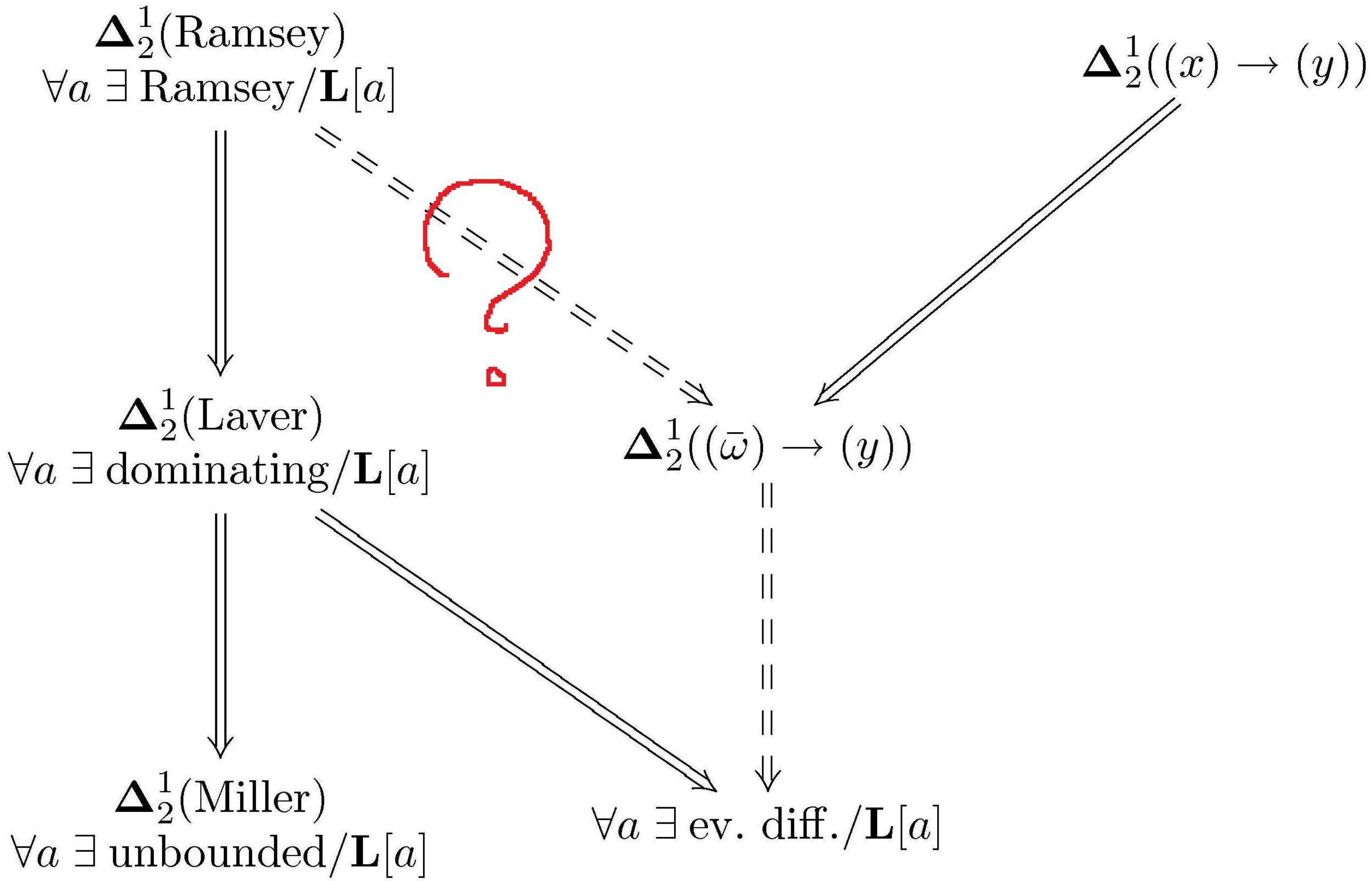

A model for $\mathbf{\Delta}^1_2((x) \to (y))$

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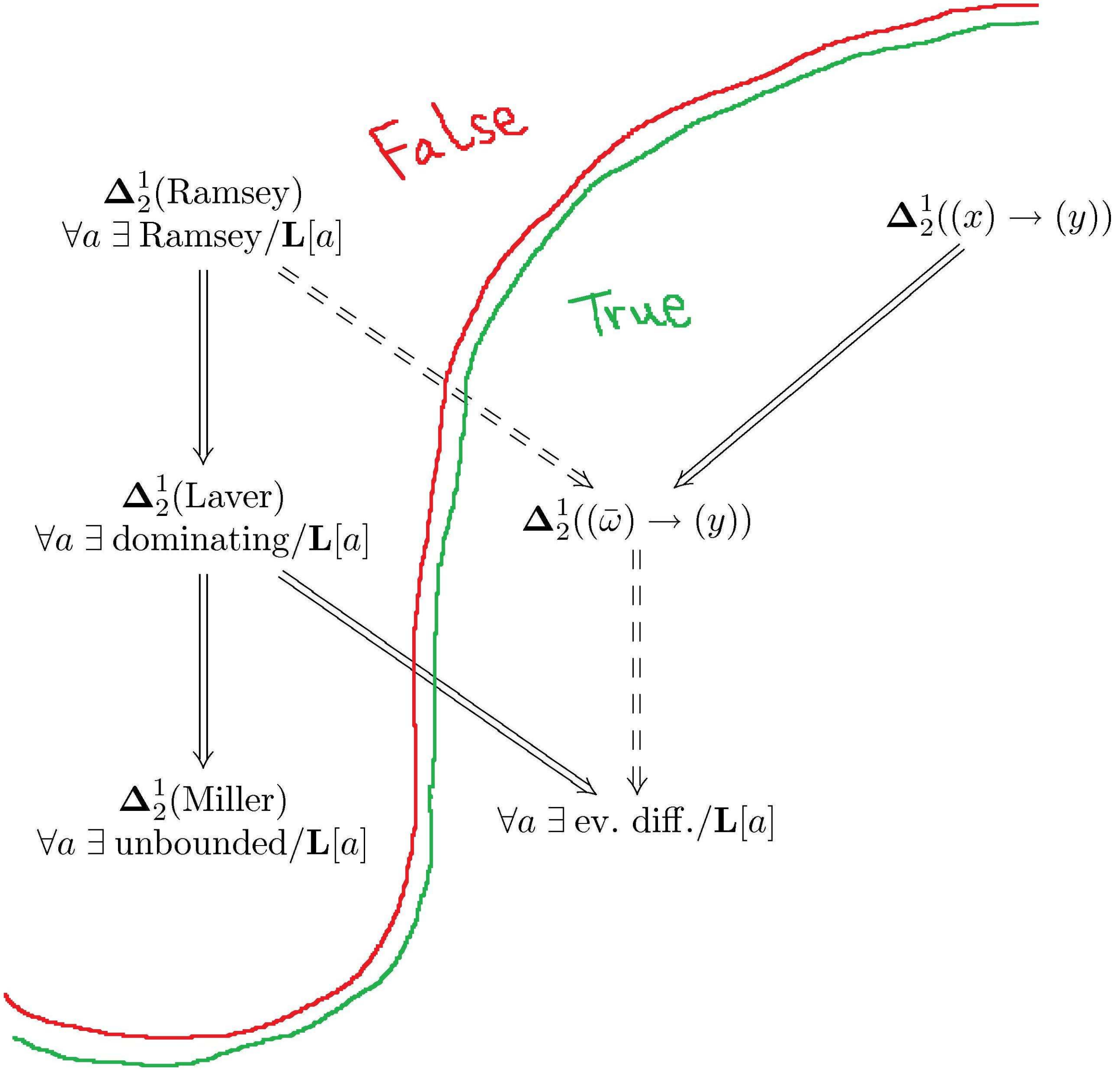

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An ω^ω-bounding forcing doesn't add unbounded reals, hence an ω_1-iteration produces a model of $\neg\mathbf{\Delta}^1_2$(Miller).

So we must find an ω^ω-bounding forcing s.t. an ω_1-iteration produces $\mathbf{\Delta}^1_2((x) \to (y))$.

The forcing notion

"Almost" theorem. There is such a forcing notion.

Idea. Let $\mathbb{P}_{\mathrm{FUT}}$ (for "fat uniform tree forcing") consist of finitely branching, uniform trees, equiv. conditions of the form (s, H) where $s \in \omega^{<\omega}$ and $H : \omega \to [\omega]^{<\omega}$.

Some lower bounds are required to make sure that the trees are sufficiently branching beyond the stem (i.e., $\|H\|$ is sufficiently increasing).

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3. An ω_1-iteration with countable support of $\mathbb{P}_{\mathrm{FUT}}$, starting from **L**, yields a model in which $\mathbf{\Delta}^1_2((x) \to (y))$ holds.

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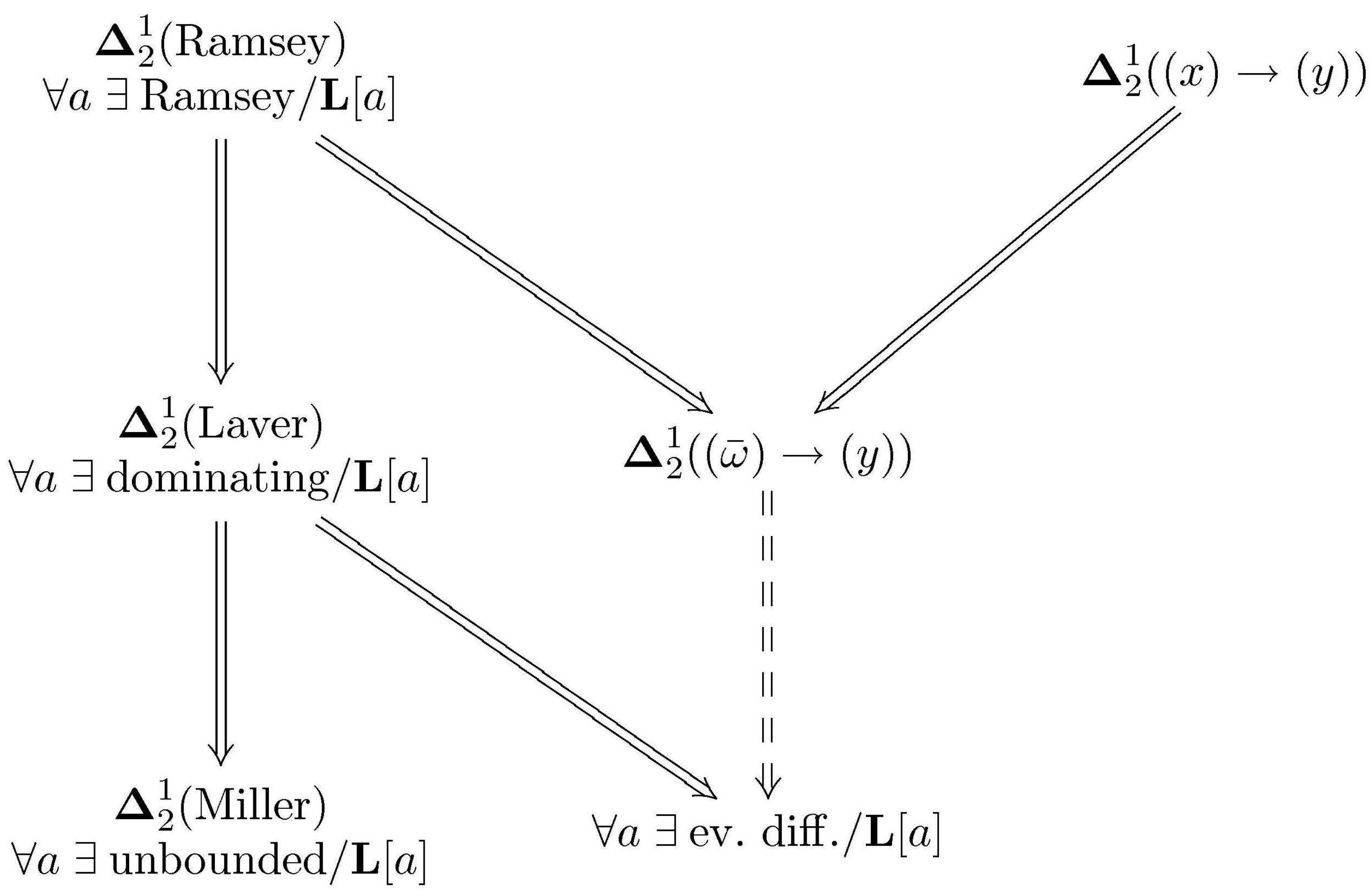

Future work

Still open:

1. Is the implication $\mathbf{\Delta}^1_2((\bar{\omega}) \to (y)) \Rightarrow \forall a\, \exists \text{ev. diff.}/\mathbf{L}[a]$ strict? Conjecture: yes.

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2. Relationship with Lebesgue measurability.

- $\mathbf{\Delta}^1_2(\text{Lebesgue}) \not\Rightarrow \mathbf{\Delta}^1_2((\bar{\omega}) \to (y))$ in the random model?
- $\mathbf{\Delta}^1_2((\bar{\omega}) \to (y)) \not\Rightarrow \mathbf{\Delta}^1_2(\text{Lebesgue})$ in the $\mathbb{P}_{\text{FUT}}$ model?

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3. What happens at the $\mathbf{\Sigma}^1_2$-level?

   Basic question: are $\mathbf{\Sigma}^1_2((\bar{\omega}) \to (y))$ and $\mathbf{\Delta}^1_2((\bar{\omega}) \to (y))$ equivalent?

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