Polarized partition properties on the second level of the projective hierarchy

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Introduction: Regularity properties and the projective hierarchy.
Regularity Properties

Regularity properties for sets of reals

(e.g. Lebesgue measurability, Baire property, Ramsey property, Bernstein property)
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More regularity on $\Delta^1_2/\Sigma^1_2$-level \( \sim \) \( L \) gets smaller
Examples

- $\Delta^1_2$(Lebesgue) $\iff \forall a \exists$ random-generic/$L[a]$
- $\Delta^1_2$(Baire Property) $\iff \forall a \exists$ Cohen-generic/$L[a]$
Examples

- \( \Delta^1_2(\text{Lebesgue}) \iff \forall a \exists \text{random-generic}/L[a] \)
- \( \Delta^1_2(\text{Baire Property}) \iff \forall a \exists \text{Cohen-generic}/L[a] \)
- \( \Delta^1_2(\text{Ramsey}) \iff \forall a \exists \text{Ramsey real }/L[a] \)
- \( \Delta^1_2(\text{Laver}) \iff \forall a \exists \text{dominating real }/L[a] \)
- \( \Delta^1_2(\text{Miller}) \iff \forall a \exists \text{unbounded real }/L[a] \)
- \( \Delta^1_2(\text{Sacks}) \iff \forall a \exists \text{real } \notin L[a] \)
More Examples

- $\Sigma^1_2$ (Lebesgue) $\iff \forall a \exists \text{measure-one set of random-generics}/L[a]
- $\Sigma^1_2$ (Baire Property) $\iff \forall a \exists \text{comeager set of Cohen-generics}/L[a]$
More Examples

- \( \Sigma^1_2(\text{Lebesgue}) \iff \forall a \exists \text{ measure-one set of random-generics}/L[a] \)
- \( \Sigma^1_2(\text{Baire Property}) \iff \forall a \exists \text{ comeager set of Cohen-generics}/L[a] \)
- \( \Sigma^1_2(\text{Ramsey}) \iff \Delta^1_2(\text{Ramsey}) \)
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Implications and non-implications

Given two regularity properties $\text{Reg}_1$ and $\text{Reg}_2$ we are interested in:

$$\Gamma(\text{Reg}_1) \implies \Gamma'(\text{Reg}_2)?$$

for $\Gamma, \Gamma' \in \{\Delta^1_2, \Sigma^1_2\}$
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- Positive answer: find a ZFC-proof
- Negative answer: find a model $M$ s.t. $M \models \Gamma(\text{Reg}_1)$ but $M \models \neg \Gamma'(\text{Reg}_2)$
Introduction

Implications and non-implications

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What has been established so far?
Diagram of implications

\[ \Sigma_2^1(E) = \Sigma_2^1(D) \]
\[ \Sigma_2^1(B) = \Delta_2^1(A) \]
\[ \Sigma_2^1(R) = \Delta_2^1(R) \]
\[ \Sigma_2^1(C) = \Delta_2^1(D) \]
\[ \Delta_2^1(E) \]
\[ \Delta_2^1(B) \]
\[ \Sigma_2^1(L) = \Delta_2^1(L) \]
\[ \Delta_2^1(C) \]
\[ \Sigma_2^1(V) \]
\[ \text{ev. diff.} \]
\[ \Sigma_2^1(M) = \Delta_2^1(M) \]
\[ \Delta_2^1(V) \]
\[ \Sigma_2^1(S) = \Delta_2^1(S) \]

Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*
Polarized partition properties.
Polarized partitions

We work in $\omega^\omega$. Letters $H, J, \ldots$ stand for infinite sequences of finite subsets of $\omega$, i.e. $H : \omega \rightarrow [\omega]^<\omega$. Use abbreviation: $[H] = \prod_{i \in \omega} H(i)$. 
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**Definition (unbounded polarized partition)**

A set $A \subseteq \omega^\omega$ satisfies the property $\left( \begin{array}{c} \omega \\ \ldots \end{array} \right) \rightarrow \left( \begin{array}{c} m_0 \\ m_1 \\ \ldots \end{array} \right)$ if

$$\exists H \text{ s.t. } \forall i \ (|H(i)| = m_i) \text{ and } [H] \subseteq A \text{ or } [H] \cap A = \emptyset$$
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A set $A \subseteq \omega^\omega$ satisfies the property $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ if

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and $n_1, n_2, \ldots$ are recursive in $m_1, m_2, \ldots$. 
Some facts

Polarized partition properties have been studied by Henle, Llopis, DiPrisco, Todorčević and Zapletal.
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Easy observations:

1. In order for $(\vec{n} \rightarrow \vec{m})$ to hold even for very simple sets, $\vec{n} \gg \vec{m}$.
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Easy observations:

1. In order for $(\vec{n} \rightarrow \vec{m})$ to hold even for very simple sets, $\vec{n} \gg \vec{m}$.
2. $\Gamma(\vec{n} \rightarrow \vec{m}) \implies \Gamma(\vec{\omega} \rightarrow \vec{m})$.
3. $\Gamma(\vec{\omega} \rightarrow \vec{m}) \iff \Gamma(\vec{\omega} \rightarrow \vec{m}')$, for all $\vec{m}, \vec{m} \geq 2$.
   - If $\Gamma(\vec{n} \rightarrow \vec{m})$, then for every other $\vec{m}'$ there is $\vec{n}'$ such that $\Gamma(\vec{n}' \rightarrow \vec{m}')$
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   - If \(\Gamma(\vec{n} \to \vec{m})\), then for every other \(\vec{m}'\) there is \(\vec{n}'\) such that \(\Gamma(\vec{n}' \to \vec{m}')\).

From now on, use generic notations \((\vec{\omega} \to \vec{m})\) and \((\vec{n} \to \vec{m})\).
Which sets satisfy this property?

In [DiPrisco & Todorcevic, 2003]:

- $(\vec{\omega} \rightarrow \vec{m})$ and $(\vec{n} \rightarrow \vec{m})$ hold for analytic sets.
- Explicit bounds $\vec{n}$ computed from $\vec{m}$ (using Ackermann-like function).
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So, what about $\Delta^1_2/\Sigma^1_2(\vec{\omega} \to \vec{m})$ and $\Delta^1_2/\Sigma^1_2(\vec{n} \to \vec{m})$?
$\Delta^1_2$-level: easy results.
Upper and lower bounds

Theorem

\[ \Delta^1_2(Ramsey) \implies \Delta^1_2(\bar{\omega} \rightarrow \bar{m}). \]
Upper and lower bounds

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Theorem (Brendle)
If \( \Delta^1_2(\bar{\omega} \to \bar{m}) \) then \( \forall a \) there is an eventually different real over \( L[a] \).
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Theorem
\[ \Delta_2^1(Ramsey) \iff \Delta_2^1(\vec{\omega} \to \vec{m}). \]

Theorem (Brendle)

If \( \Delta_2^1(\vec{\omega} \to \vec{m}) \) then \( \forall a \) there is an eventually different real over \( L[a] \).

Proof.

Assume otherwise and use the canonical \( \Delta_2^1(a) \)-well-ordering of \( L[a] \) to construct a counterexample.
Diagram of implications

\[ \Delta^1_2(\text{Ramsey}) \]
\[ \forall a \exists \text{Ramsey}/L[a] \]

\[ \Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \]

\[ \Delta^1_2(\bar{m} \rightarrow \bar{m}) \]

\[ \forall a \exists \text{ev. diff.}/L[a] \]
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**Question:** which implications cannot be reversed?
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\[ \Delta^1_2(n \to \bar{m}) \]

Question: which implications cannot be reversed?
Theorem (Brendle-Kh)

Let \( V \) be obtained by an \( \omega_1 \)-iteration of Mathias forcing beginning from \( L \). Then \( \Delta^1_2(\text{Ramsey}) \) holds whereas \( \Delta^1_2(\vec{n} \to \vec{m}) \) fails.
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\[ \Delta^1_2(\text{Ramsey}) \]
\[ \forall a \exists \text{Ramsey} / L[a] \]

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\[ \Delta^1_2(\bar{m} \rightarrow m) \]

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False

True
Mathias model

**Theorem (Brendle-Kh)**

Let $V$ be obtained by an $\omega_1$-iteration of Mathias forcing beginning from $L$. Then $\Delta^1_2$(Ramsey) holds whereas $\Delta^1_2(\vec{n} \to \vec{m})$ fails.
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Proof.

Use the fact that Mathias forcing satisfies the Laver property.
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\[ \Delta_2^1(\tilde{\omega} \rightarrow \tilde{m}) \]

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\[ \Delta_2^1(\vec{n} \rightarrow \vec{m}) \]
$\Delta^1_2$-level: creature forcing.
The non-implication

Theorem (Brendle-Kh)

There is a model in which $\Delta^1_2(\vec{n} \rightarrow \vec{m})$ holds but $\Delta^1_2(\text{Miller})$ fails. (i.e. there are no unbounded reals).
Diagram of implications
The non-implication

Theorem (Brendle-Kh)

There is a model in which $\Delta^1_2(n \to m)$ holds but $\Delta^1_2$ (Miller) fails.

(i.e. there are **no unbounded reals**).
The non-implication

**Theorem (Brendle-Kh)**

*There is a model in which $\Delta^1_2(\vec{n} \rightarrow \vec{m})$ holds but $\Delta^1_2(\text{Miller})$ fails. (i.e. there are no unbounded reals).*

Look for a forcing notion which is:
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Look for a forcing notion which is:

1. Proper and $\omega^\omega$-bounding—for all $\dot{x}$ there is a $y$ in the ground model and a $p$ s.t. $p \Vdash \forall n (\dot{x}(n) \leq \dot{y}(n))$. 
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2. An $\omega_1$-iteration beginning from $L$ yields a model where $\Delta^1_2(\vec{n} \rightarrow \vec{m})$ holds.
Creature forcing

Such a forcing notion exists!
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A certain kind of creature forcing, due to [Kellner-Shelah, 2009] and [Shelah-Zapletal, unpublished]. We shall refer to it as $\mathbb{P}_{KSZ}$. 

Definition

At each $n$, a small $\epsilon_n$ is given, and we build a "mini-forcing" $\mathbb{P}_n$:

Let $X(n) \in \omega$ be a 'large' upper bound. $\mathbb{P}_n$ consists of 'conditions' of the form $(c, k)$ with $c \subseteq X(n)$ and $k \in \omega$ such that $\log_3(|c|) \geq k + 1$.

Let $a_n := 2^{1/\epsilon_n}$. Define

$\text{norm}_n(c, k) := \log_3(a_n)(\log_3(|c|) - k)$

for each $(c, k) \in \mathbb{P}_n$.

If $X(n)$ is sufficiently large, then $\exists (c, k) \in \mathbb{P}_n$ s.t. $\text{norm}_n(c, k) \geq n$.

[To be precise: $X(n)$ must be larger than $2^{((2^{1/\epsilon_n})^n)}$.]
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  - $(c', k') \leq_n (c, k)$ iff $c' \subseteq c$ and $k' \geq k$. 
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[To be precise: $X(n)$ must be larger than $2^{((2^{1/\epsilon_n})^n)}$]
Definition (...continued)

Now let $P_{KSZ}$ consist of conditions $p$ such that:

There is stem $(p) \in \omega^{<\omega}$ and $\forall n \geq |stem(p)|: p(n) \in P_n$,

$$\lim_{n \to \infty} \text{norm}_n(p(n)) = \infty.$$  

$p' \leq p$ iff $stem(p') \supseteq stem(p)$

For $n$ with $|stem(p')| \leq n < |stem(p')|$: $p'(n) \in$ first coordinate of $p(n)$

For $n \geq |stem(p')|$: $p'(n) \leq np(n)$

Remark: $P_{KSZ}$ adds a generic real $x_G := \bigcup \{ stem(p) | p \in G \}$, but the generic filter is not determined from the generic real in the usual fashion and $P_{KSZ}$ is not in general representable as $B(\omega\omega)/I$ for a $\sigma$-ideal $I$. 

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Remark: $\mathbb{P}_{KSZ}$ adds a generic real $x \in G = \bigcup \{\text{stem}(p) | p \in G\}$, but the generic filter is not determined from the generic real in the usual fashion and $\mathbb{P}_{KSZ}$ is not in general representable as $\mathbb{B}(\omega^\omega)/\mathbb{I}$ for a $\sigma$-ideal $\mathbb{I}$. 

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Proper and $\omega^\omega$-bounding

Theorem (Kellner-Shelah, Shelah-Zapletal)

If $\mathbb{P}_{KSZ}$ is as above, and moreover $\forall n \left( \epsilon_n \leq \frac{1}{n \cdot \prod_{j<n} X(j)} \right)$, then $\mathbb{P}_{KSZ}$ is proper and $\omega^\omega$-bounding.
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**Proof.**

Show that each component $\mathbb{P}_n$ of $\mathbb{P}_{KSZ}$ satisfies two properties from the general theory of creature forcings: “$\epsilon_n$-bigness” and “$\epsilon_n$-halving”. □
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Proof.

Show that each component $P_n$ of $P_{KSZ}$ satisfies two properties from the general theory of creature forcings: “$\varepsilon_n$-bigness” and “$\varepsilon_n$-halving”.

Remark: $X(n)$ is a function of $\varepsilon_n$, and $\varepsilon_n$ is a function of $X(m)$ for $m < n$. So we have to define them inductively.
Forcing $\Delta^1_2(\vec{n} \to \vec{m})$

**Theorem (Brendle-Kh)**

An $\omega_1$-iteration of $\mathbb{P}_{\text{KSZ}}$, starting from $L$, gives a model in which $\Delta^1_2(\vec{n} \to \vec{m})$ holds but $\Delta^1_2(\text{Miller})$ fails.
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The bounds “$\vec{n}$” have been explicitly computed beforehand: they are the $X(n)$’s from the definition of $\mathbb{P}_{KSZ}$. 
Diagram of implications

$\Delta_2^1\text{-level: creature forcing}$

$\Delta_2^1(\text{Ramsey})$
$\forall a \exists \text{Ramsey}/L[a]$

$\Delta_2^1(\text{Laver})$
$\forall a \exists \text{dominating}/L[a]$

$\Delta_2^1(\text{Miller})$
$\forall a \exists \text{unbounded}/L[a]$

$\Delta_2^1(\vec{\omega} \rightarrow \vec{m})$

$\Delta_2^1(\vec{n} \rightarrow \vec{m})$

$\forall a \exists \text{ev. diff.}/L[a]$
Open questions for $\Delta^1_2$
Open questions

1. Is the implication $\Delta^1_2(\vec{\omega} \rightarrow \vec{m}) \implies \exists$ ev. diff. reals strict?
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1. Is the implication $\Delta^1_2(\bar{\omega} \rightarrow \bar{m}) \implies \exists$ ev. diff. reals strict?

**Conjecture:** Yes, $\Delta^1_2(\bar{\omega} \rightarrow \bar{m})$ fails in the Random model.
Open questions for $\Delta^1_2$

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4. Same for $(\vec{n} \rightarrow \vec{m})$. 
The $\Sigma^1_2$-level
Forcing $\Sigma^1_2(\vec{n} \to \vec{m})$

Can we extend the result about $\mathbb{P}_{KSZ}$ to $\Sigma^1_2$?
Forcing $\Sigma_2^1(\vec{n} \rightarrow \vec{m})$

Can we extend the result about $\mathbb{P}_{KSZ}$ to $\Sigma_2^1$? 

Not a priori, since $\mathbb{P}_{KSZ}$ only adds one generic real.
Forcing $\Sigma^1_2(\vec{n} \rightarrow \vec{m})$

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In [DiPrisco & Todorčević, 2003] a forcing is defined which adds a **generic product** $H_G$ satisfying what we will call the “clopification property”:
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For all Borel sets $B$ in the ground model, $B \cap [H_G]$ is relatively clopen in $[H_G]$. 
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Theorem (Brendle-Kh)

An $\omega_1$-iteration of any (proper) forcing notion with the clopification property, starting from $L$, gives a model where $\Sigma^1_2(\vec{n} \rightarrow \vec{m})$ holds.
Problem with using the DiPrisco-Todorčević forcing: difficult to see whether it is $\omega^\omega$-bounding or not.
Forcing $\Sigma^1_2(\vec{n} \rightarrow \vec{m})$: continued

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So instead, we combine elements of the DiPrisco-Todorčević forcing with $\mathbb{P}_{KSZ}$,
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The $\Sigma^1_2$-level

Forcing $\Sigma^1_2(\vec{n} \rightarrow \vec{m})$: continued

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**Corollary**

*There is a model where $\Sigma^1_2(\vec{n} \rightarrow \vec{m})$ holds but $\Sigma^1_2(Miller)$ fails.*
Thank you!

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