Suslin Proper Forcing and Regularity Properties.

Yurii Khomskii
KGRC, Vienna

joint with Vera Fischer and Sy Friedman

Final INFTY Conference, Bonn, 4 March 2014

Purpose of this talk

- Recently, in joint work with Vera Fischer and Sy Friedman, we made some progress in separating various regularity properties on the $\mathbf{\Delta}^1_3$, $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_4$ levels.
- We constructed models by iterating "definable" forcing.
- I will talk about the methods involved.

Sample result

$$
\begin{tikzcd}
\mathbf{\Delta}^1_3(\mathbb{B}) \arrow[r, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{V}) \arrow[rr, Rightarrow] & & \mathbf{\Delta}^1_3(\mathbb{S}) \\
\mathbf{\Delta}^1_3(\mathbb{R}) \arrow[r, Rightarrow] \arrow[ur, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
 & & \mathbf{\Delta}^1_3(\mathbb{C}) \arrow[u, Rightarrow] \arrow[uul, Rightarrow] &
\end{tikzcd}
$$

$\mathbb{C}$ = Baire property; $\mathbb{B}$ = Lebesgue measure; $\mathbb{S}$ = Sacks-measurability; $\mathbb{M}$ = Miller-measurability; $\mathbb{L}$ = Laver-measurability; $\mathbb{V}$ = Silver measurability; $\mathbb{R}$ = Ramsey property.

Theorem (Fischer-Friedman-Kh)

Each constellation of "true"/"false" assignments (18 possibilities) to the above statements not contradicting this diagram, is consistent r.t. ZFC or ZFC + inaccessible.

Sacks forcing

I will present just **one** of the methods, using **Sacks forcing** as a canonical example.

Definition

Sacks forcing $\mathbb{S}$ is the partial order of perfect trees on $2^{<\omega}$ ordered by inclusion.

Sacks-measurability

Definition

$A \subseteq 2^\omega$ is **Sacks-measurable** iff there is a perfect tree $T \subseteq 2^{<\omega}$ such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

The usual definition is "below any Sacks-condition S there is a $T \leq S$ ...", but this is equivalent for sufficiently closed pointclasses $\boldsymbol{\Gamma}$.

Sacks-measurability depends on the complexity of A

Theorem (Bernstein 1908)

There exists a non Sacks-measurable set.

Proof.

Enumerate perfect trees $\{T_\alpha \mid \alpha < 2^{\aleph_0}\}$ and “diagonalize” (Bernstein set). $\square$

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Theorem (Suslin 1917)

All analytic sets are Sacks-measurable.

Modern proof.

Let $A = \{x \mid \phi(x)\}$. Let $\dot{x}_G$ be the name for the Sacks-generic real, and let T be a Sacks-condition deciding $\phi(\dot{x}_G)$, w.l.o.g. $T \Vdash \phi(\dot{x}_G)$. Let $M \prec \mathcal{H}_\theta$ be a **countable elementary submodel** with $\mathbb{S}, T \in M$. Using a **properness argument** find $S \leq T$ such that all $x \in [S]$ are Sacks-generic over M. So for all $x \in [S]$, $M[x] \models \phi(x)$, and by $\mathbf{\Sigma}^1_1$**-absoluteness** $\phi(x)$. Therefore $[T] \subseteq A$. $\square$

Beyond $\mathbf{\Sigma}^1_1$

Theorem (Gödel 1938)

$L \models \neg \mathbf{\Delta}^1_2(\mathbb{S})$.

Proof.

Again diagonalize against all perfect trees, but use the **$\mathbf{\Sigma}^1_2$-good wellorder of the reals** of L. $\square$

Beyond $\mathbf{\Sigma}^1_1$

Theorem (Folklore)

$V^{\mathbb{S}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{S})$.

Proof.

Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$, w.l.o.g. parameters in V. The statement $\forall x\,(\phi(x) \leftrightarrow \neg\psi(x))$ is $\mathbf{\Pi}^1_3$ hence downward absolute between $V^{\mathbb{S}_{\omega_1}}$ and $V^{\mathbb{S}}$. In V find Sacks-condition T forcing $\phi(\dot{x}_G)$ or $\psi(\dot{x}_G)$, and proceed as before (and use **upwards $\mathbf{\Sigma}^1_2$-absoluteness** from $M[x]$ to $V^{\mathbb{S}_{\omega_1}}$). $\square$

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Remark: It is not hard to do better and obtain $V^{\mathbb{S}_{\omega_1}} \models \boldsymbol{\Sigma}^1_2(\mathbb{S})$.

Beyond Σ^1_2

Question

Can we use similar methods to obtain $\mathbf{\Delta}^1_3(\mathbb{S})$, $\mathbf{\Sigma}^1_3(\mathbb{S})$ etc.?

Problems:

1. We used **Shoenfield absoluteness** and **$\mathbf{\Sigma}^1_1$-absoluteness** for countable models.
2. Using coding techniques (e.g. "almost disjoint coding") one can force a $\mathbf{\Sigma}^1_3$-good wellorder of the reals over L, contradicting $\mathbf{\Delta}^1_3(\mathbb{S})$.

This suggests that the **definability** of the forcing iteration plays a role.

Properness Without Elementaricity

- Recall: $\mathbb{P}$ is **proper** iff for every $M \prec \mathcal{H}_\theta$ with $\mathbb{P} \in M$ and every $p \in \mathbb{P} \cap M$, there is $q \leq p$ which is $(M, \mathbb{P})$-generic, i.e.

$$q \Vdash \text{“}M[\dot{G}] \text{ is a } (\mathbb{P} \cap M)\text{-generic extension of } M\text{”}.$$

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- Idea: replace $M \prec \mathcal{H}_\theta$ by **any** countable transitive model M of (a sufficient fragment of) ZFC.
- But what does “$\mathbb{P} \cap M$” etc. mean when M is not elementary?

Suslin proper forcing

Definition

A forcing $\mathbb{P}$ is **Suslin** if elements of $\mathbb{P}$ are (coded by) reals and "$p \in \mathbb{P}$", "$p \leq q$" and "$p \perp q$" are $\mathbf{\Sigma}^1_1$ relations.

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If $\mathbb{P}$ is Suslin and M is any countable model containing the parameters defining $\mathbb{P}$, then $\mathbb{P}^M$ refers to the **interpretation** of $\mathbb{P}$ within M.

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Definition

A forcing notion $\mathbb{P}$ is **Suslin proper** if it is Suslin and for **any** countable transitive model M containing the parameters of $\mathbb{P}$, and every $p \in \mathbb{P}^M$, there is $q \leq p$ which is $(M, \mathbb{P})$-generic, i.e.,

$$q \Vdash \text{"}M[\dot{G}] \text{ is a } \mathbb{P}^M\text{-generic extension of } M\text{"}.$$

Suslin$^+$ proper forcing

Unfortunately, many standard forcing notions (in particular Sacks, Miller and Laver) are not **exactly** Suslin, because $\perp$ is only $\mathbf{\Pi}^1_1$ but not $\mathbf{\Sigma}^1_1$.

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Solution: (Shelah; Goldstern) Replace "Suslin" by "Suslin$^+$", where we don't require $\perp$ to be $\mathbf{\Sigma}^1_1$. Instead, we make sure that there is an "effective" version of being an $(M, \mathbb{P})$-generic condition.

Technically, require that there exists a $\mathbf{\Sigma}^1_2$, $(\omega+1)$-place relation $\mathrm{epd}(p_i, q)$ such that if $\mathrm{epd}(p_i, q)$ holds then $\{p_i \mid i < \omega\}$ is predense below q and use epd to define an effectively $(M, \mathbb{P})$-generic condition.

Remarks about Suslin$^+$

Remarks:

1. Suslin ccc $\Rightarrow$ Suslin proper $\Rightarrow$ Suslin$^+$ proper $\Rightarrow$ proper.
2. All standard definable forcings used in the theory of the reals which are known to be proper, are actually Suslin$^+$ proper.

   Jakob Kellner, Preserving non-null with Suslin$^+$ forcings, Arch. Math. Logic (2006) 45:649–664.

Complexity of the forcing relation

Lemma

Let $\mathbb{P}$ be Suslin$^+$ proper and τ a nice $\mathbb{P}$-name for a real. Then for any $\mathbf{\Pi}^1_n$-formula θ, the statement "$p \Vdash_{\mathbb{P}} \theta(\tau)$" is also $\mathbf{\Pi}^1_n$, for all $n \geq 2$.

(*Here we consider τ as coded by a real*).

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Proof.

Induction on n, base case $n = 2$.

Let θ be $\mathbf{\Pi}^1_2$. Then $p \Vdash \theta(\tau)$ iff for all countable transitive models M containing p, τ and all parameters of $\mathbb{P}$, $M \models p \Vdash \theta(\tau)$. This statement is $\mathbf{\Pi}^1_2$.

The rest follows by induction. □

Iteration of Suslin$^+$ proper forcing

Judah, Shelah and Goldstern developed the theory of **iterations** of Suslin and Suslin$^+$ proper forcings.

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If $\mathbb{P}_{\omega_1} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a countable support iteration **of length** ω_1, where every iterand is Suslin$^+$ proper, then:

1. $\mathbb{P}_\alpha$-names for reals, and conditions $p \in \mathbb{P}_\alpha$, are coded by reals.
2. "$p \in \mathbb{P}_\alpha$" and "$p \leq_\alpha q$" are $\mathbf{\Pi}^1_2$.
3. If θ is a $\mathbf{\Pi}^1_n$ formula for $n \geq 2$, $p \in \mathbb{P}_\alpha$ and τ a nice $\mathbb{P}_\alpha$-name for a real, then "$p \Vdash_\alpha \theta(\tau)$" is $\mathbf{\Pi}^1_n$.

Main Result

Theorem (Fischer-Friedman-Kh)

$V^{\mathbb{S}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{S})$.

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Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be $\mathbf{\Delta}^1_3$, w.l.o.g. parameters in V.

Let x_0 be first Sacks-generic over V. W.l.o.g. $V[G_{\omega_1}] \models \phi(x_0)$. Then $V[G_{\omega_1}] \models \exists y \theta(x_0, y)$ for some $\mathbf{\Pi}^1_2$ formula θ. By properness, there is $\alpha < \omega_1$ such that $y \in V[G_\alpha]$, and by Shoenfield absoluteness $V[G_\alpha] \models \theta(x_0, y)$. In V, let p be a $\mathbb{S}_\alpha$-condition and τ a nice $\mathbb{S}_\alpha$-name for a real, such that

$$p \Vdash_\alpha \theta(\dot{x}_{G(0)}, \tau).$$

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Proof.

Then $V[x_0] \models$ "if we force with the remainder $\mathbb{S}_{1,\alpha} \cong \mathbb{S}_\alpha$ along p interpreted using x_0, then $\theta(\check{x}_0, \tau[x_0])$ will hold".

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Formally, let $\tilde{\theta}(x)$ be a conjunction of the following statements:

- "$p[x]$ is an $\mathbb{S}_\alpha$-condition",
- "$\tau[x]$ is a nice $\mathbb{S}_\alpha$-name for a real", and
- $p[x] \Vdash_\alpha \theta(\check{x}, \tau[x])$.

Then $V[x_0] \models \tilde{\theta}(x_0)$. Moreover, $\tilde{\theta}$ is $\mathbf{\Pi}^1_2$.

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Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

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Proof.

In V we have $p(0) \Vdash_{\mathbb{S}} \tilde{\theta}(\dot{x}_{G(0)})$.

Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t. $\forall x \in [T]\,(x$ is $\mathbb{S}$-generic over $V)$.

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Let $\Theta(T)$ abbreviate "$\forall x \in [T]\, \tilde{\theta}(x)$". This is $\boldsymbol{\Pi}^1_2$.

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Argue in $V[x_0]$. It is known that if you add a Sacks-real you add a perfect set of Sacks-reals, even below any perfect set. So there is a $T \leq p(0)$ s.t.
$\forall x \in [T]\,(x$ is $\mathbb{S}$-generic over $V)$.
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$\forall x \in [T]\,\tilde{\theta}(x)$.

Let $\Theta(T)$ abbreviate "$\forall x \in [T]\,\tilde{\theta}(x)$". This is $\mathbf{\Pi}^1_2$.

Then $V[G_\beta] \models \Theta(T)$ for $1 \leq \beta < \omega_1$.

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Theorem (Fischer-Friedman-Kh)

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Proof.

We claim $V[G_{\omega_1}] \models [T] \subseteq A$.

Main result

Theorem (Fischer-Friedman-Kh)

$V^{\mathbb{S}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{S})$.

Proof.

We claim $V[G_{\omega_1}] \models [T] \subseteq A$.

Pick $z \in [T]$, let $\beta < \omega_1$ be such that $z \in V[G_\beta]$. Since $V[G_\beta] \models \Theta(T)$ in particular $V[G_\beta] \models \tilde{\theta}(z)$, so in particular

$$V[G_\beta] \models p[z] \Vdash_{\mathbb{S}_\alpha} \theta(\check{z}, \tau[z]).$$

By genericity we may assume β is sufficiently large so that $p[z]$ is in the generic.

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It follows that $V[G_{\beta+\alpha}] \models \theta(z, \tau[z][G_{[\beta+1,\beta+\alpha)}])$, hence $V[G_{\beta+\alpha}] \models \phi(z)$, and by upwards-absoluteness, $V[G_{\omega_1}] \models \phi(z)$. $\square$

Generalizations

Let $\mathbb{P}$ be a forcing whose conditions are trees on 2^ω or ω^ω ordered by inclusion.

Definition

A is $\mathbb{P}$-**measurable** iff there is $T \in \mathbb{P}$ such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

The only essential property of Sacks forcing we used is: if you add a Sacks-real you add a perfect set of Sacks-reals.

Amoeba and Quasi-amoeba

Definition

Let $\mathbb{P}$ be a tree-like forcing notion, and $\mathbb{AP}$ another forcing. We say that

1. $\mathbb{AP}$ is a **quasi-amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, in $V[G]$ there is a $q \leq p$ such that

$$V[G] \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

2. $\mathbb{AP}$ is an **amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, in $V[G]$ there is a $q \leq p$ such that for any larger model $W \supseteq V[G]$,

$$W \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

Examples

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Examples:

1. Sacks forcing is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
2. Miller forcing is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
3. Laver forcing is **not** a quasi-amoeba for itself (Brendle 1998), but there are amoebas for Laver.
4. Mathias forcing is an amoeba for itself.

General theorem

Theorem (Fischer-Friedman-Kh)

Suppose $\mathbb{P}$ is a tree-like forcing, $\mathbb{AP}$ a **quasi-amoeba** *for $\mathbb{P}$, and both $\mathbb{P}$ and $\mathbb{AP}$ are* **Suslin$^+$ proper***. Then $V^{(\mathbb{P}*\mathbb{AP})_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P})$.*

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Applications:

- $\mathbb{M}$ = Miller forcing. $V^{\mathbb{M}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{M})$.
- $\mathbb{R}$ = Mathias forcing. $V^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_3(\text{Ramsey})$ (Judah-Shelah).
- $\mathbb{L}$ = Laver forcing, $\mathbb{AL}$ = "amoeba for Laver". $V^{(\mathbb{L}*\mathbb{AL})_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{L})$.

Mixing forcings

You can also mix other things in the iteration: e.g. if $\mathbb{P}$ is a tree-like forcing and $\mathbb{AP}$ a $\mathbb{P}$-quasi-amoeba, you can use any ω_1-iteration where $\mathbb{P}$ and $\mathbb{AP}$ appear cofinally often, assuming that the iteration is sufficiently "repetitive", and all iterands are **Suslin$^+$ proper** (this is essential, since otherwise we could mix coding and obtain a model with $\mathbf{\Sigma}^1_3$-good wellorder!)

Other generalizations

Zapletal's idealized framework: forcing with $\mathcal{B}(\omega^\omega)/I$ for a σ-ideal I on the reals.

Definition

A is I-**measurable** iff there is Borel set $B \notin I$ such that $B \subseteq A$ or $B \cap A = \varnothing$.

In this case, "amoeba" and "quasi-amoeba" means "adding a Borel I-positive set of $\mathcal{B}(\omega^\omega)/I$-generic reals". All the above results still apply, assuming $\mathcal{B}(\omega^\omega)/I$ is Suslin$^+$ proper.

Application

$$\begin{tikzcd}
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Each constellation of "true"/"false" assignments (18 possibilities) to the above statements not contradicting this diagram, is consistent r.t. ZFC or ZFC + inaccessible.

Thank you!

Yurii Khomskii
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And thanks to INFTY for supporting me!