

# LOWER BOUNDS ON THE ARITHMETIC SELF-INTERSECTION NUMBER OF THE RELATIVE DUALIZING SHEAF ON ARITHMETIC SURFACES

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ABSTRACT. We give an explicitly computable lower bound for the arithmetic self-intersection number  $\bar{\omega}^2$  of the dualizing sheaf on a large class of arithmetic surfaces. If some technical conditions are satisfied, then this lower bound is positive. In particular, these technical conditions are always satisfied for minimal arithmetic surfaces with simple multiplicities and at least one reducible fiber, but we have also used our techniques to obtain lower bounds for some arithmetic surfaces with non-reduced fibers.

## 1. INTRODUCTION

Let  $K$  be a number field, let  $\mathcal{O}_K$  denote the ring of integers of  $K$ . Let  $X/K$  denote a smooth, projective, geometrically irreducible curve of genus  $g > 1$ , let  $\pi : \mathcal{X} \rightarrow \text{Spec}(\mathcal{O}_K)$  be a proper regular model of  $X$  and let  $\bar{\omega} = \bar{\omega}_{\mathcal{X}}$  denote the relative dualizing sheaf on  $\mathcal{X}$  over  $\text{Spec}(\mathcal{O}_K)$ , equipped with the Arakelov metric. The arithmetic self-intersection  $\bar{\omega}^2$  is one of the most fundamental objects in arithmetic intersection theory; see for instance [20] for a discussion. In this note we show how to effectively compute lower bounds on  $\bar{\omega}^2$  in many situations including, but not limited to, semistable  $\mathcal{X}$ .

To each  $\mathbb{Q}$ -divisor  $D \in \text{Div}_{\mathbb{Q}}(X) = \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  of degree one we attach in Definition 4.5 a hermitian line bundle  $\overline{\mathcal{L}}_D$ . We show that the height  $h_{\overline{\mathcal{L}}_D}(\cdot)$  with respect to  $\overline{\mathcal{L}}_D$  is closely related to the Néron-Tate height induced by the embedding  $j_D$  of  $X$  into its Jacobian via  $D$ . More precisely, we define a certain finite set  $T(\mathcal{X})$  of closed points on  $\mathcal{X}$  and prove the following result in Section 4, where we write  $D_{\mathcal{X}}$  for the Zariski closure in  $\mathcal{X}$  of an irreducible divisor  $D \in \text{Div}(X)$  and extend this to  $\text{Div}_{\mathbb{Q}}(X)$  by linearity.

**Theorem 1.1.** *Suppose that  $D \in \text{Div}_{\mathbb{Q}}(X)$  has degree one and  $\text{supp}(D_{\mathcal{X}}) \cap T(\mathcal{X}) = \emptyset$ . Then, if  $E = \sum_{j=1}^d (P_j)$  is an irreducible divisor on  $X$ , where  $P_j \in X(\bar{K})$ , and  $\text{supp}(E_{\mathcal{X}}) \cap T(\mathcal{X}) = \emptyset$ , we have*

$$h_{\overline{\mathcal{L}}_D}(E) = \frac{1}{d} \sum_{j=1}^d h_{\text{NT}}(j_D(P_j)) \geq 0.$$

*In particular, we have  $h_{\overline{\mathcal{L}}_D}(P) = h_{\text{NT}}(j_D(P))$  for all  $P \in X(K)$ .*

If  $\mathcal{X}$  is semistable, then  $T(\mathcal{X})$  is simply the set of singular points on the special fibers of  $\mathcal{X}$ . Note that in the proof of [23, Theorem 5.6], Zhang proves an analogue (with  $T(\mathcal{X}) = \emptyset$ ) of Theorem 1.1 in the language of his admissible intersection theory. Since we want to be able to compute lower bound on  $\bar{\omega}^2$  for non-semistable  $\mathcal{X}$ , we cannot use the admissible theory and have to work with hermitian line bundles throughout.

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*Date:* July 15, 2013.

The second author was supported by DFG-grant KU 2359/2-1.

In order to use Theorem 1.1 to derive a lower bound on  $\bar{\omega}^2$ , we follow Zhang's approach from [23]. The idea is to show that under certain conditions, we have  $\overline{\mathcal{L}_D}^2 \geq 0$ . If these conditions are satisfied and  $(2g-2)D$  is a canonical  $\mathbb{Q}$ -divisor on  $X$ , then we can relate  $\overline{\mathcal{L}_D}^2$  to  $\bar{\omega}^2$  and use this inequality to obtain lower bounds on  $\bar{\omega}^2$ .

As in Zhang's theory (cf. [22, Theorem 6.5]), the crucial condition on  $\overline{\mathcal{L}_D}$  is relative semi-positivity, where we call a hermitian line bundle *relatively semipositive* if its restriction to every irreducible vertical divisor has nonnegative arithmetic degree (see Definition 3.2). The proof of the following result is similiar to the proof of [22, Theorem 6.5], but rather more complicated.

**Proposition 1.2.** *If  $\overline{\mathcal{L}_D}$  is relatively semipositive and  $D_{\mathcal{X}} \cap T(\mathcal{X}) = \emptyset$ , then we have  $\overline{\mathcal{L}_D}^2 \geq 0$ .*

In Section 2 we locally define certain vertical divisors  $V_D$  and  $U_D$  attached to  $D$ ; they are the main ingredients in the construction of  $\overline{\mathcal{L}_D}$ , see Definition 4.5. Moreover, we set

$$\beta_D = \frac{1-g}{g} \mathcal{O}(2V_D + U_D)^2 + 2(\bar{\omega} \cdot \mathcal{O}(U_D)).$$

**Theorem 1.3.** *Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  be a  $\mathbb{Q}$ -divisor such that  $(2g-2)D$  is a canonical  $\mathbb{Q}$ -divisor on  $X$  and such that  $D_{\mathcal{X}}$  satisfies  $D_{\mathcal{X}} \cap T(\mathcal{X}) = \emptyset$ . If the hermitian line bundle  $\overline{\mathcal{L}_D}$  is relatively semipositive, then we have*

$$\bar{\omega}^2 \geq \beta_D.$$

Note that a divisor  $D$  as in Theorem 1.3 always exists. In order to derive a nontrivial lower bound on  $\bar{\omega}^2$  from Theorem 1.3 for a given  $\mathcal{X}$ , we need to show that  $\overline{\mathcal{L}_D}$  is relatively semipositive and that for some choice of  $D$  as in the statement of the theorem, we have  $\beta_D \geq 0$ .

**Theorem 1.4.** *If  $\mathcal{X}$  is minimal and all special fibers of  $\mathcal{X}$  are reduced, then the following are satisfied for every divisor  $D \in \text{Div}_{\mathbb{Q}}(X)$  of degree one:*

- (i)  $\overline{\mathcal{L}_D}$  is relatively semipositive;
- (ii)  $\beta := \beta_D$  does not depend on the choice of  $D$ ;
- (iii)  $\beta \geq 0$ , with equality if and only if all special fibers of  $\mathcal{X}$  are irreducible.

The proof of Theorem 1.4 essentially follows from a sequence of local lemmas proved in Section 2. We provide an explicit formula for  $\beta$  in Lemma 2.14, making it very simple to compute  $\beta$  for a given minimal model  $\mathcal{X}$  with reduced fibers. As an immediate corollary we recover the following result from [23], [17] and [4].

**Corollary 1.5.** *If  $\mathcal{X}$  is semistable and minimal and has at least one reducible fiber, then there is an effectively computable positive lower bound on  $\bar{\omega}^2$ .*

In the semistable case lower bounds on  $\bar{\omega}^2$  can be derived by means of the admissible intersection theory due to Zhang, cf. [23, Theorem 5.5]. This method requires the computation of admissible Green's functions on the reduction graphs of the special fibers of  $\mathcal{X}$  and has been employed by Abbes-Ullmo [1] to find lower bounds for certain modular curves (see also Subsection 6.1), but such an approach does not work for non-semistable arithmetic surfaces. The positivity of  $\bar{\omega}^2$  for non-semistable  $\mathcal{X}$  with at least one reducible has been proven by Sun in [19]. However, his result is often not suitable for explicit computations of such bounds in practice, since it requires computing a global semistable model over an extension of  $K$ .

We believe that for  $D$  as in Theorem 1.3,  $\beta_D$  is a lower bound on  $\bar{\omega}^2$  for all minimal arithmetic surfaces, even those with non-reduced fibers. Indeed, if we are given a minimal  $\mathcal{X}$  having components of multiplicity  $> 1$ , we can still check whether Theorem 1.3 is applicable. As an example, we prove that the conditions of Theorem 1.3 are satisfied for the minimal regular model  $\mathcal{F}_p^{\min}$  of the Fermat curve of prime exponent  $p > 3$  over the field of  $p$ -th cyclotomic numbers and that the resulting lower bound is positive. This does not follow from Theorem 1.4, since the irreducible components of  $\mathcal{F}_p^{\min}$  need not have multiplicity one.

**Theorem 1.6.** *The arithmetic self-intersection  $\bar{\omega}^2$  of the relative dualizing sheaf on  $\mathcal{F}_p^{\min}$  satisfies*

$$\bar{\omega}^2 \geq p \log p + \mathcal{O}(\log p).$$

A more precise statement is provided in Theorem 6.6.

The paper is organized as follows: In Section 2 we define the divisors  $V_D$  and  $U_D$  locally and prove that they have certain properties with respect to the intersection multiplicity. We then switch to a global perspective in Section 3, where we prove some general results on hermitian line bundles. Section 4 contains the definition of  $\overline{\mathcal{L}}_D$  and the proof of Theorem 1.1. The results of Sections 2, 3 and 4 are then used in Section 5 to prove Proposition 1.2 and Theorems 1.3 and 1.4. At the end of that section, we also discuss a possible application of our results to the effective Bogomolov conjecture. In Section 6 we first use Theorem 1.4 to prove an asymptotic lower bound for  $\bar{\omega}^2$  on minimal regular models of modular curves  $X_1(N)$  for certain  $N$ . Finally, we use Theorem 1.3 to prove Theorem 1.6; here we also compare the resulting lower bound to the upper bound computed by Curilla and the first author in [8].

We would like to thank Ariyan Javanpeykar and David Holmes for a careful reading of the manuscript and many helpful suggestions. We also thank Zubeyir Cinkir for helpful advice on the proof of Lemma 2.11 and Christian Curilla for drawing Figure 1.

## 2. INTERSECTION PROPERTIES OF CERTAIN VERTICAL DIVISORS

Let  $\mathcal{O}$  be a strictly Henselian discrete valuation ring with field of fractions  $K$ . Let  $\mathcal{X}_s$  be the special fiber of a proper regular model  $\mathcal{X}/\mathcal{O}$  of a smooth projective geometrically irreducible curve  $X/K$  of genus  $g > 1$ . In this section we define certain vertical divisors  $V_D, U_D$  with support in the special fiber  $\mathcal{X}_s$  attached to  $\mathbb{Q}$ -divisors  $D \in \text{Div}_{\mathbb{Q}}(X)$  and study their properties.

Suppose that, as a divisor on  $\mathcal{X}$ , the special fiber  $\mathcal{X}_s$  is given by  $\mathcal{X}_s = \sum_{i=1}^r b_i \Gamma_i$ , where  $\{\Gamma_1, \dots, \Gamma_r\}$  is the set of irreducible components of  $\mathcal{X}_s$  and the  $b_i$  are positive integers. We fix a canonical divisor  $\mathcal{K}$  on  $\mathcal{X}$ , and set

$$a_i = (\Gamma_i \cdot \mathcal{K})$$

for  $i \in \{1, \dots, r\}$ , where  $(\cdot)$  is the rational-valued intersection multiplicity on  $\mathcal{X}$ . Note that by the adjunction formula [13, Theorem 9.1.37], we have

$$a_i = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2,$$

where  $p_a(\Gamma_i)$  is the arithmetic genus of  $\Gamma_i$ . Given a nonzero  $\mathbb{Q}$ -divisor  $D \in \text{Div}_{\mathbb{Q}}(X)$ , we denote the Zariski closure of  $D$  in  $\mathcal{X}$  by  $D_{\mathcal{X}}$ .

**Proposition 2.1.** *For every  $D \in \text{Div}_{\mathbb{Q}}(X)$  there exists a vertical  $\mathbb{Q}$ -divisor  $V_D \in \text{Div}_{\mathbb{Q}}(\mathcal{X})$ , which is unique up to addition of rational multiples of  $\mathcal{X}_s$ , such that*

$$(D_{\mathcal{X}} + V_D \cdot \Gamma_i) = \frac{\deg(D)}{2g-2} a_i$$

for all  $i \in \{1, \dots, r\}$ . Moreover, the assignment

$$D \mapsto V_D \pmod{\mathcal{X}_s}$$

is linear in  $D$ .

*Proof.* The assignment

$$E \mapsto \left( \left( \frac{\deg(D)}{2g-2} \mathcal{K} - D_{\mathcal{X}} \right) \cdot E \right)$$

defines a linear map on  $Z^1(\mathcal{X}_s)_{\mathbb{Q}} = Z^1(\mathcal{X}_s) \otimes_{\mathbb{Z}} \mathbb{Q}$ . By the non-degeneracy of the intersection pairing on  $Z^1(\mathcal{X}_s)_{\mathbb{Q}}$  modulo the entire fiber, this map is representable by a cycle  $V_D \in Z^1(\mathcal{X}_s)_{\mathbb{Q}}$ . As this assignment is also a linear map in  $D$ , the two claims follow immediately.  $\square$

Proposition 2.1 implies that we can extend any  $\mathbb{Q}$ -divisor  $D$  on  $X$  to a  $\mathbb{Q}$ -divisor on  $\mathcal{X}$  which satisfies the adjunction formula up to a factor  $\deg(D)$ . We can define a local pairing on coprime divisors  $E_1, E_2 \in \text{Div}_{\mathbb{Q}}(X)$  by

$$(1) \quad [E_1, E_2] = (E_{1,\mathcal{X}} + V_{E_1} \cdot E_{2,\mathcal{X}} + V_{E_2}).$$

**Corollary 2.2.** *The pairing  $[E_1, E_2]$  extends the local Néron pairing (see [12, §III.5]) to divisors of arbitrary degree.*

To our knowledge, the pairing  $[\cdot, \cdot]$  is the first extension of the local Néron pairing to divisors of arbitrary degree that is not based on the reduction graph as in [5] or [23].

**Corollary 2.3.** *Suppose that  $D_l \in \text{Div}_{\mathbb{Q}}(X)$  satisfies  $(D_{l,\mathcal{X}} \cdot \Gamma_i) = \delta_{il}$  for all  $i \in \{1, \dots, r\}$ . Then there exists a vertical divisor  $V_l := V_{D_l}$  such that*

$$(2) \quad (V_l \cdot \Gamma_i) = a'_i - \delta_{il},$$

where  $a'_i = \frac{1}{2g-2} a_i$ .

Several applications will rely explicit formulas for the  $V_l$ . We define a matrix  $M = (m_{ij})_{i,j}$  as minus the intersection matrix of  $\mathcal{X}_s$ :

$$m_{i,j} = -(b_i \Gamma_i \cdot b_j \Gamma_j)$$

Let  $M^+ = (n_{ij})_{i,j}$  denote the Moore-Penrose pseudoinverse of  $M$ , cf. [7, §3]. For fixed  $l \in \{1, \dots, r\}$  we define a vector

$$c_l = (c_{l1}, \dots, c_{lr})^t = -M^+ w_l,$$

where

$$w_l = (w_{l1}, \dots, w_{lr})^t, \quad w_{lj} = b_j a'_j - \delta_{lj};$$

here  $\delta_{il}$  is the Kronecker delta.

**Proposition 2.4.** *If  $l \in \{1, \dots, r\}$ , then a divisor  $V_l$  satisfying (2) exists. Moreover, we have*

$$V_l = \sum_{i=1}^r b_i c_{li} \Gamma_i.$$

*Proof.* It follows from [2, Corollary 9.1.10] that a  $\mathbb{Q}$ -divisor  $D_l$  satisfying  $(D_{l,\mathcal{X}} \cdot \Gamma_i) = \delta_{il}$  exists for all  $i \in \{1, \dots, r\}$ . The formula for  $V_l$  is an immediate consequence of the relations

$$MM^+M = M \quad \text{and} \quad M^+MM^+ = M^+.$$

$\square$

*Definition 2.5.* If  $D \in \text{Div}_{\mathbb{Q}}(X)$  has degree  $d > 0$ , then we define a vertical  $\mathbb{Q}$ -divisor  $U_D$  on  $\mathcal{X}$  associated to  $D$  as follows: For all  $i \in \{1, \dots, r\}$  we set

$$\gamma_{D,i} = \frac{1}{d} (V_D^2 - (V_D - dV_i))^2$$

and define

$$U_D = \sum_{i=1}^r \gamma_{D,i} \Gamma_i.$$

Our main motivation for this definition is the following formula for the intersection of  $U_D$  with horizontal divisors. It will play a crucial part in the proof of Theorem 1.1. If  $D$  has degree 0, then we write

$$(3) \quad \Phi_{\mathcal{X}}(D) := V_D$$

in accordance with the classical literature (see for instance [12, Theorem III.3.6]).

**Proposition 2.6.** *Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  have degree  $d > 0$  and let  $E = \sum_{j=1}^e (P_j)$  be a nontrivial effective divisor on  $X$ , where  $P_j \in X(K)$ . Then we have*

$$d(E_{\mathcal{X}} \cdot U_D) = eV_D^2 - \sum_{j=1}^e \Phi_{\mathcal{X}}(dP_j - D)^2.$$

Moreover, the association  $D \mapsto U_D$  is linear in  $D$ .

*Proof.* For every  $j \in \{1, \dots, e\}$  there is an index  $i_j \in \{1, \dots, r\}$  such that the section corresponding to  $P_j$  intersects  $\Gamma_{i_j}$  and does not intersect any other component. Therefore

$$(4) \quad \sum_{j=1}^e \Phi_{\mathcal{X}}(dP_j - D)^2 = \sum_{j=1}^e \left( d^2 V_{i_j}^2 - 2d(V_{i_j} \cdot V_D) + V_D^2 \right) = -d \sum_{j=1}^e \gamma_{D,i_j} + eV_D^2.$$

The first assertion follows from (4) and

$$(E_{\mathcal{X}} \cdot U_D) = \sum_{j=1}^e (P_{j,\mathcal{X}} \cdot U_D) = \sum_{j=1}^e \gamma_{D,i_j}.$$

The second assertion is trivial. □

We are now ready to define a local version of what will be our lower bound on  $\bar{\omega}^2$ .

*Definition 2.7.* If  $D \in \text{Div}_{\mathbb{Q}}(X)$  is a divisor of degree 1, we define

$$\beta_D = \frac{1-g}{g} (2V_D + U_D)^2 + 2(\mathcal{K} \cdot U_D).$$

*Example 2.8.* Suppose that the special fiber of  $\mathcal{X}$  consists of two irreducible components  $\Gamma_1$  and  $\Gamma_2$  of multiplicity 1 and identical arithmetic genus  $p_a$  which intersect transversally in  $s \geq 1$  points. Let  $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$ . Then it is easy to see that we can take  $V_D = 0$  and

$$V_1 = \frac{1}{4s} \Gamma_1 - \frac{1}{4s} \Gamma_2 = -V_2.$$

This leads to

$$U_{D_1} = -\frac{1}{4s}\Gamma_1 + \frac{3}{4s}\Gamma_2 = -U_{D_2} \quad \text{and}$$

$$U_D = \frac{1}{4s}\Gamma_1 + \frac{1}{4s}\Gamma_2.$$

A simple computation reveals

$$\beta_D = \frac{1}{2s}(s + 2p_a - 2).$$

In order to show that (a global version of)  $\beta_D$  indeed provides a non-trivial lower bound for  $\bar{\omega}^2$  in many situations, we first need to prove some further intersection-theoretic properties of  $U_D$ . To this end, we define a metrized graph  $\mathcal{G}_{\mathcal{X}}$  as follows: The vertex set of  $\mathcal{G}_{\mathcal{X}}$  is given by  $\{\Gamma_1, \dots, \Gamma_r\}$ . There are no self-loops or multiple edges; two vertices  $\Gamma_i$  and  $\Gamma_j$  are connected by an edge if and only if  $m_{ij} \neq 0$ , in which case the length of the edge is  $-1/m_{ij}$ . We also need some facts about the matrix  $M$  and its pseudoinverse  $M^+$ .

**Lemma 2.9.** *The following properties are satisfied:*

- (i) Both  $M$  and  $M^+$  are symmetric and positive semidefinite.
- (ii) We have  $\sum_{j=1}^r m_{ij} = \sum_{j=1}^r n_{ij} = 0$  for all  $i \in \{1, \dots, r\}$ .
- (iii) We have  $\sum_{j=1}^r n_{ij} m_{jk} = -\frac{1}{r} + \delta_{kl}$  for all  $i, k \in \{1, \dots, r\}$ .
- (iv) We have  $n_{ii} - \sum_{j,k} n_{ij} n_{kk} m_{jk} = \frac{\text{Tr}(M^+)}{r}$  for all  $i \in \{1, \dots, r\}$ .
- (v)  $M$  is the discrete Laplacian matrix associated to  $\mathcal{G}_{\mathcal{X}}$ .

*Proof.* These properties are proved in [7]. □

*Remark 2.10.* Note that when  $\mathcal{X}$  is minimal and semistable,  $\mathcal{G}_{\mathcal{X}}$  need not coincide with the reduction graph  $R(X)$  associated to  $X$  in [5] and [23]. For instance, suppose that  $\mathcal{X}_s$  is given by two curves  $\Gamma_1$  and  $\Gamma_2$  intersecting transversally in  $n$  points. In this case,  $R(X)$  is the banana graph with  $n$  edges of length 1, whereas  $\mathcal{G}_{\mathcal{X}}$  is the complete graph with two vertices which are connected by a single edge of length  $1/n$ .

From now on, we suppose that  $D \in \text{Div}_{\mathbb{Q}}(X)$  has degree one. For  $i \in \{1, \dots, r\}$  we set

$$v_i(D) = (b_i \Gamma_i \cdot D_{\mathcal{X}}) \quad \text{and}$$

$$w_i(D) = b_i a'_i - v_i(D) = \frac{b_i a_i}{2g - 2} - v_i(D).$$

**Lemma 2.11.** *Let  $i \in \{1, \dots, r\}$ .*

(a) *We have*

$$(U_D \cdot \Gamma_i) = -\sum_{j=1}^r n_{jj} m_{ij} + 2v_i(D) - \frac{2}{r}.$$

(b) *If the special fiber of  $\mathcal{X}$  is reduced, then*

$$(\mathcal{K} \cdot \Gamma_i) + 2(D_{\mathcal{X}} \cdot \Gamma_i) - (U_D \cdot \Gamma_i)$$

*is independent of  $D$  and nonnegative.*

*Proof.* Assertion (a) is an easy computation using Lemma 2.9:

$$\begin{aligned}
 (U_D \cdot \Gamma_i) &= - \sum_{l=1}^r \sum_{j=1}^r c_{lj} (w_{lj} - 2w_j(D)) m_{li} \\
 &= - \sum_{l=1}^r \left( \sum_{j=1}^r c_{lj} (2v_j(D) - a'_j - \delta_{lj}) \right) m_{li} \\
 &= - \sum_{l=1}^r \sum_{j=1}^r \sum_{k=1}^r (2n_{kj} a'_k v_j(D) - n_{kj} a'_k a'_j - n_{kj} a'_k \delta_{lj} - 2n_{kj} \delta_{kl} v_j(D) \\
 &\quad + n_{kj} \delta_{kl} a'_j + n_{kj} \delta_{kl} \delta_{lj}) m_{li} \\
 &= - \sum_{j=1}^r n_{jj} m_{ij} + 2 \sum_{l=1}^r \sum_{j=1}^r n_{lj} v_j(D) m_{li} \\
 &= - \sum_{j=1}^r n_{jj} m_{ij} + 2 \sum_{j=1}^r v_j(D) \left( \delta_{ij} - \frac{1}{r} \right) \\
 &= - \sum_{j=1}^r n_{jj} m_{ij} + 2v_i(D) - \frac{2}{r}.
 \end{aligned}$$

Now we turn to assertion (b) of the lemma and compute, using (a) and the adjunction formula:

$$(5) \quad (\mathcal{K} \cdot \Gamma_i) + 2(D_{\mathcal{X}} \cdot \Gamma_i) - (U_D \cdot \Gamma_i) = m_{ii} + 2p_a(\Gamma_i) - 2 + \sum_{j=1}^r n_{jj} m_{ij} + \frac{2}{r}$$

We deduce the first part of (b) and furthermore:

$$(6) \quad \sum_j n_{jj} m_{ij} = \sum_j (n_{jj} + n_{ii} - 2n_{ij}) m_{ij} - \frac{2}{r} + 2$$

Note that by [7, Lemma 4.1] we have

$$n_{jj} + n_{ii} - 2n_{ij} = r(\Gamma_i, \Gamma_j),$$

where  $r(\Gamma_i, \Gamma_j)$  is the effective resistance between the nodes  $\Gamma_i$  and  $\Gamma_j$  if we consider the metrized graph  $\mathcal{G}_{\mathcal{X}}$  as a resistive electric circuit, where the resistance along an edge  $e$  is given by the length  $\ell(e)$ .

Hence, using (5) and (6), it suffices to show

$$(7) \quad m_{ii} + \sum_j r(\Gamma_i, \Gamma_j) m_{ij} \geq 0$$

in order to prove assertion (b). But we can rewrite the left hand side of (7) as

$$\sum_{j \neq i} m_{ij} (r(\Gamma_i, \Gamma_j) - 1)$$

because of Lemma 2.9 (ii). A component  $\Gamma_j$  can only contribute a negative summand to this sum if  $m_{ij} \neq 0$ , which means that the nodes on  $\mathcal{G}_{\mathcal{X}}$  corresponding to  $\Gamma_i$  and  $\Gamma_j$  are connected by an edge  $e$  of length  $\ell(e) = -\frac{1}{m_{ij}} \leq 1$ . But in this case the effective resistance  $r(\Gamma_i, \Gamma_j)$  is

bounded from above by  $\ell(e)$ . Hence all terms in the sum are nonnegative, proving (7) and thus the lemma.  $\square$

If  $\mathcal{X}_s$  is reduced, then we can give a formula for the intersection of  $U_D$  with a canonical divisor. This result will be important in order to show that for reduced  $\mathcal{X}_s$ , our lower bound  $\beta_D$  does not depend on  $D$  and that  $\beta_D$  is nonnegative if  $\mathcal{X}$  is also minimal.

**Lemma 2.12.** *Suppose that the special fiber of  $\mathcal{X}$  is reduced. Then we have*

$$(U_D \cdot \mathcal{K}) = - \sum_{i=1}^r V_i^2 a_i.$$

*If, furthermore,  $\mathcal{X}$  is minimal, then  $(U_D \cdot \mathcal{K})$  is nonnegative.*

*Proof.* Since  $D \mapsto U_D$  is linear in  $D$  by Proposition 2.6, we may assume that we have  $D = D_l$  for some  $l \in \{1, \dots, r\}$ , that is,  $(D \cdot \Gamma_j) = \delta_{lj}$  for all  $j \in \{1, \dots, r\}$ .

By definition of  $U_D$ , we have

$$(U_D \cdot \mathcal{K}) = - \sum_{i=1}^r V_i^2 a_i + 2 \sum_{i=1}^r (V_i \cdot V_l) a_i,$$

so we have to show that

$$(8) \quad \sum_{i=1}^r (V_i \cdot V_l) a_i = 0.$$

Note that

$$(V_i \cdot V_l) = \sum_{j=1}^r c_{lj} (V_i \cdot \Gamma_j) = \sum_{j=1}^r c_{lj} w_{ij},$$

so that, using  $c_l^t = M^+ w_l^t$ , we find

$$\sum_{i=1}^r (V_i \cdot V_l) a_i = -(2g - 2) w_l^t M^+ \sum_{i=1}^r a'_i w_i.$$

Therefore the proof of (8) follows from the fact that for each  $j \in \{1, \dots, r\}$  we have

$$\sum_{i=1}^r w_{ij} a'_i = a'_j \sum_{i=1}^r a'_i - a'_j = 0,$$

since  $\sum_{i=1}^r a'_i = 1$ .

If  $\mathcal{X}$  is minimal, then the adjunction formula implies that

$$a_i = (\mathcal{K} \cdot \Gamma_i) = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2 \geq 0.$$

for all  $i$ , which completes the proof of the lemma.  $\square$

The following result can be deduced immediately from Lemma 2.12.

**Corollary 2.13.** *Suppose that  $\mathcal{X}$  is minimal with reduced special fiber. Then  $\beta_D$  is nonnegative.*

It is natural to ask whether  $\beta_D$  depends on  $D$ . We will now show that this is not the case when  $\mathcal{X}$  has reduced special fiber; furthermore, we will provide a rather explicit formula for  $\beta_D$ .



**Lemma 2.14.** *If  $\mathcal{X}$  has reduced special fiber, then  $\beta = \beta_D$  is independent of  $D$ . More precisely, we have the following formula for  $\beta$ :*

$$\beta = \frac{4(g-1)}{gr} \text{Tr}(M^+) + \frac{g-1}{g} \sum_{i=1}^r \sum_{j=1}^r n_{ii} n_{jj} m_{ij} + \frac{2(g-1)}{g} \sum_{i=1}^r a_i n_{ii} - \frac{1}{g} \sum_{i=1}^r \sum_{j=1}^r a_i a_j n_{ij}.$$

*Proof.* The proof is essentially a straightforward, but tedious computation using the properties of  $M$  and  $M^+$  listed in Lemma 2.9, so we do not present all details. Suppose that  $\mathcal{X}$  has reduced special fiber.

We first give an expression for  $(K \cdot U_D)$ . By Lemma 2.12, we have

$$(K \cdot U_D) = - \sum_{i=1}^r V_i^2 a_i.$$

A simple computation shows that for a fixed  $i \in \{1, \dots, r\}$  we have

$$V_i^2 = 2 \sum_j a'_j n_{ij} - n_{ii} - \sum_{j,k} a'_j a'_k n_{jk};$$

using Lemma 2.9 and  $\sum_i a'_i = 1$ , this implies

$$(9) \quad (K \cdot U_D) = (2g-2) \left( \sum_i a'_i n_{ii} - \sum_{i,j} a'_i a'_j n_{ij} \right).$$

Now we rewrite  $(U_D + 2V_D)^2$ . From the definition of  $V_D$  we get

$$(10) \quad V_D^2 = 2 \sum_{i,j} a'_i v_j(D) n_{ij} - \sum_{i,j} a'_i a'_j n_{ij} - \sum_{i,j} v_i(D) v_j(D) n_{ij}.$$

Next we compute, using Lemma 2.9 and omitting details:

$$\begin{aligned} (V_D \cdot U_D) &= \sum_i (2(V_D \cdot V_i) - V_i^2) w_i(D) \\ &= \sum_{i,j,k} a'_j a'_k v_i(D) n_{jk} - 2 \sum_{i,j,k} a'_k v_i(D) v_j(D) n_{jk} + 2 \sum_{i,j} v_i(D) v_j(D) n_{ij} \\ &\quad + \sum_{i,j,k} a'_i a'_j a'_k n_{jk} - 2 \sum_{ij} a'_i a'_j n_{ij} + \sum_i a'_i n_{ii} - \sum_i v_i(D) n_{ii} \\ (11) \quad &= 2 \sum_{i,j} v_i(D) v_j(D) + \sum_i a'_i n_{ii} - 2 \sum_{i,j} a'_i v_j(D) n_{ij} - \sum_i v_i(D) n_{ii}. \end{aligned}$$

The computation of  $U_D^2$  more complicated than the previous one, so we only provide a rough sketch:

$$\begin{aligned} U_D^2 &= -4 \sum_{i,j} (V_D \cdot V_i)(V_D \cdot V_j) m_{ij} + 4 \sum_{i,j} (V_D \cdot V_i) V_j^2 m_{ij} - \sum_{i,j} V_i^2 V_j^2 m_{ij} \\ &= 4 \sum_{i,j,k} v_i(D) n_{ij} n_{kk} m_{jk} - \sum_{i,j} n_{ii} n_{jj} m_{ij} - 4 \sum_{i,j,k,l} v_i(D) v_l(D) n_{ij} n_{kl} m_{jk} \\ (12) \quad &= 4 \sum_{i,j,k} v_i(D) n_{ij} n_{kk} m_{jk} - \sum_{i,j} n_{ii} n_{jj} m_{ij} - 4 \sum_{i,j} v_i(D) v_j(D) n_{ij}. \end{aligned}$$

Type	$\varepsilon$	$\beta$
I	0	0
II( $a$ )	$a$	$a - 1$
III( $a$ )	$\frac{1}{6}a$	$\frac{1}{6}a - \frac{1}{6a}$
IV( $a, b$ )	$a + \frac{1}{6}b$	$a + \frac{1}{6}b - \frac{1}{6b}$
V( $a, b$ )	$\frac{1}{6}(a + b)$	$\frac{1}{6}(a + b) - \frac{1}{6a} - \frac{1}{6b}$
VI( $a, b, c$ )	$a + \frac{1}{6}(b + c)$	$a + \frac{1}{6}(b + c) - \frac{1}{6b} - \frac{1}{6c}$
VII( $a, b, c$ )	$\frac{1}{6}(a + b + c) + \frac{1}{6} \frac{abc}{ab+ac+bc}$	$\frac{1}{6}(a + b + c) + \frac{1}{6} \frac{abc}{ab+ac+bc} - \frac{a^2b+a^2c+ab^2+6abc+ac^2+b^2c+bc^2}{6(ab+ac+bc)^2}$

TABLE 1. The invariants  $\varepsilon$  and  $\beta$  in genus 2

Combining (10), (11) and (12), we find

$$(U_D + 2V_D)^2 = 4 \sum_i a'_i n_{ii} - \sum_{i,j} n_{ii} n_{jj} m_{ij} - 4 \sum_{i,j} a'_i a'_j n_{ij} + 4 \sum_{i,j,k} v_i(D) n_{ij} n_{kk} m_{jk} - 4 \sum_i v_i(D) n_{ii}.$$

Part (iv) of Lemma 2.9 implies

$$(13) \quad (U_D + 2V_D)^2 = 4 \sum_i a'_i n_{ii} - \sum_{i,j} n_{ii} n_{jj} m_{ij} - 4 \sum_{i,j} a'_i a'_j n_{ij} - \frac{4\text{Tr}(M^+)}{r},$$

which does not depend on  $D$ . The result now follows from (9) and (13).  $\square$

*Example 2.15.* Keeping the notation of Example 2.8, Lemma 2.14 immediately implies that

$$\beta = \frac{1}{2s}(s + 2p_a - 2).$$

*Remark 2.16.* Note that the first two terms in the formula for  $\beta$  given in Lemma 2.14 only depend on  $M$ , so they only depend on the combinatorial configuration of  $\mathcal{X}_s$ . The last two terms, however, do depend on the arithmetic genera of the irreducible components; more precisely, we have

$$\frac{2(g-1)}{g} \sum_{i=1}^r a_i n_{ii} - \frac{1}{g} \sum_{i=1}^r \sum_{j=1}^r a_i a_j n_{ij} = \frac{2g-2}{g} \sum_i a_i \left( n_{ii} - \sum_j a'_j n_{ij} \right).$$

Therefore, if  $\mathcal{X}$  is semistable and minimal, then  $\beta$  can be viewed as an invariant of the polarized metrized graph  $(R(X), \mathbf{q})$  associated to  $\mathcal{X}_s$ , where the polarization  $\mathbf{q}$  assigns to each component its arithmetic genus, see [6, §4].

*Remark 2.17.* It seems worthwhile to relate  $\beta$  to other invariants of  $(R(X), \mathbf{q})$ , such as Zhang's invariants  $\varepsilon$  (called  $r$  in [23]),  $\varphi$  and  $\lambda$ . See [6] for definitions of and some relations between these invariants. If  $X$  is hyperelliptic, then it would also be interesting to compare  $\beta$  to the invariant  $\chi$  studied, for instance, in [11]. Because of its potential relevance for an effective version of the Bogomolov conjecture for curves over number fields (see Remark 5.3), it is especially interesting to compare  $\beta$  to  $\varepsilon$ . We have computed  $\beta$  for all semistable reduction types of genus 2 curves. Table 1 contains the values of  $\beta$  and the values of  $\varepsilon$ , computed by de Jong, cf. [10, §2]. We find that in genus 2, we always have  $\beta \leq \varepsilon$ .

### 3. SEMIPOSITIVE HERMITIAN LINE BUNDLES

Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$  and let  $\mathcal{X}$  be a regular arithmetic surface over  $\mathcal{O}_K$  whose generic fiber  $X = \mathcal{X}_K$  is a smooth projective geometrically irreducible curve of genus  $g > 1$ . In this section we prove several general lemmas about certain hermitian line bundles on  $\mathcal{X}$ . All of these will be used in the proof of Proposition 1.2. Several results of this section are quite similar to results from [22]. We start with a number of definitions.

*Definition 3.1.* Let  $\bar{\mathcal{L}}$  be a hermitian line bundle on  $\mathcal{X}$ . If  $E$  is an irreducible effective divisor on  $X$  with Zariski closure  $\mathcal{E}$ , then the *height* of  $E$  with respect to  $\bar{\mathcal{L}}$  is defined by

$$h_{\bar{\mathcal{L}}}(\mathcal{E}) = \frac{(\bar{\mathcal{L}} \cdot \bar{\mathcal{O}}(\mathcal{E}))}{[K : \mathbb{Q}] \deg(\mathcal{E}_K)},$$

where the metric on  $\mathcal{O}(\mathcal{E})$  is admissible in the Arakelov-theoretic sense and  $(\cdot \cdot)$  is the arithmetic intersection pairing on  $\mathcal{X}$ , see for instance [18]. We extend this to arbitrary effective divisors on  $\mathcal{X}$  by linearity.

*Definition 3.2.* We say that a hermitian line bundle  $\bar{\mathcal{L}}$  is *relatively semipositive* if it has nonnegative intersection with all irreducible vertical components of  $\mathcal{X}$ . If  $\bar{\mathcal{L}}$  has nonnegative (resp. positive) intersection with all irreducible horizontal divisors on  $\mathcal{X}$ , then we call  $\bar{\mathcal{L}}$  *horizontally semipositive* (resp. *horizontally positive*).

*Definition 3.3.* Let  $\bar{\mathcal{L}}$  be a hermitian line bundle on  $\mathcal{X}$ . We call a nonzero section  $s$  of  $\bar{\mathcal{L}}$  *effective* (resp. *strictly effective*) if  $\|s\|_{\text{sup}} \leq 1$  (resp.  $\|s\|_{\text{sup}} < 1$ ). We say that  $\bar{\mathcal{L}}$  is *ample* if  $\mathcal{L}$  is ample and  $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$  has a basis consisting of strictly effective sections for  $n \gg 0$ . If this holds for  $n = 1$ , then we call  $\bar{\mathcal{L}}$  *very ample*.

**Lemma 3.4.** *Let  $\bar{\mathcal{L}}$  be a hermitian line bundle on  $\mathcal{X}$ . For any hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  and  $a, b \in \mathbb{N}$  we set*

$$\bar{\mathcal{M}}_{a,b} = \bar{\mathcal{M}}^{\otimes a} \otimes \bar{\mathcal{L}}^{\otimes b}.$$

*If  $\bar{\mathcal{L}}^2 < 0$ , then there exists no ample hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  with the following property: For all  $a, b \in \mathbb{N}$  such that  $\bar{\mathcal{M}}_{a,b}^2 > 0$ , we also have*

$$(14) \quad (\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}_{a,b}) \geq 0.$$

*Proof.* This proof is somewhat similar to the first part of the proof of [22, Theorem 6.3]. Suppose that  $\bar{\mathcal{L}}^2 < 0$  and that  $\bar{\mathcal{M}}$  is a hermitian line bundle on  $\mathcal{X}$  satisfying (14).

Since  $\bar{\mathcal{M}}$  is ample, it has positive arithmetic self-intersection by [22, Theorem 1.3]. Therefore, if  $p(t)$  denotes the polynomial

$$p(t) = \bar{\mathcal{L}}^2 + 2(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}})t + \bar{\mathcal{M}}^2 t^2,$$

then there is a positive real number  $t_0$  satisfying  $p(t_0) = 0$  and  $p(t) > 0$  for every  $t > t_0$ .

Let  $a, b \in \mathbb{N}$  such that  $a/b > t_0$ . Then we find

$$\bar{\mathcal{M}}_{a,b}^2 = b^2 p(a/b) > 0.$$

By (14), we know that

$$\frac{1}{b} (\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}_{a,b}) = \bar{\mathcal{L}}^2 + \frac{a}{b} (\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}) \geq 0.$$

In particular, our assumption that  $\bar{\mathcal{L}}^2 < 0$  implies

$$(15) \quad (\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}) > 0,$$

and also, since  $a/b$  can be arbitrary close to  $t_0$ ,

$$(16) \quad \bar{\mathcal{L}}^2 + (\bar{\mathcal{L}} \cdot \bar{\mathcal{M}})t_0 \geq 0.$$

Now we can derive a contradiction as in the proof of [22, Theorem 6.3]. Namely, combining  $p(t_0) = 0$  and (16) implies

$$(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}})t_0 + \bar{\mathcal{M}}^2 t_0^2 \leq 0.$$

But using  $\bar{\mathcal{M}}^2 > 0$  and (15), we see that this is impossible.  $\square$

The following result provides us with a method to show that two hermitian line bundles on  $\mathcal{X}$  have nonnegative intersection.

**Lemma 3.5.** *Let  $\bar{\mathcal{L}}$  and  $\bar{\mathcal{M}}$  be hermitian line bundles on  $\mathcal{X}$ . If  $\bar{\mathcal{M}}$  has an effective global section  $s$  such that  $h_{\bar{\mathcal{L}}}(\text{div}(s)^{\text{hor}}) \geq 0$  and  $\bar{\mathcal{L}}$  is relatively semipositive, then  $(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}) \geq 0$ .*

*Proof.* According to [3, §3.2.2], we have

$$(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}) = \frac{(\mathcal{O}(\text{div}(s)^{\text{ver}}) \cdot \bar{\mathcal{L}})}{[K : \mathbb{Q}]} + h_{\bar{\mathcal{L}}}(\text{div}(s)^{\text{hor}}) - \int \log \|s\|_{\text{sup}} c_1(\bar{\mathcal{L}}).$$

Since  $\text{div}(s)^{\text{ver}}$  is effective, the claim follows.  $\square$

Suppose we want to show that a hermitian line bundle  $\bar{\mathcal{L}}$  on  $\mathcal{X}$  satisfies  $\bar{\mathcal{L}}^2 \geq 0$ . By Lemma 3.4, it suffices to find some ample hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  such that under the assumption  $\bar{\mathcal{L}}^2 < 0$  we have  $(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}_{a,b}) \geq 0$  whenever  $\bar{\mathcal{M}}_{a,b}^2 > 0$ .

If  $\bar{\mathcal{L}}$  is horizontally semipositive, then we can take any ample  $\bar{\mathcal{M}}$  and any effective section  $s$  of  $\bar{\mathcal{M}}_{a,b}^{\otimes n}$  (which exists for  $n \gg 0$  by [22, Theorem 2.1], see [22, §8]) and apply Lemma 3.5. However, in the proof of Proposition 1.2 we will apply Lemma 3.5 to a hermitian line bundle  $\bar{\mathcal{L}} = \bar{\mathcal{L}}_D$  which is not in general horizontally semipositive, but only satisfies  $h_{\bar{\mathcal{L}}_D}(E) \geq 0$  if the Zariski closure  $E_{\mathcal{X}}$  avoids a certain finite set of points, so we have to be more careful with our choice of  $s$ .

Lemma 3.6 below tells us that, under the hypothesis  $\bar{\mathcal{L}}^2 < 0$ , we can find  $\bar{\mathcal{M}}$  such that for  $n \gg 0$  there are many effective sections of  $\bar{\mathcal{M}}_{a,b}$  whenever  $\bar{\mathcal{M}}_{a,b}^2 > 0$ . Intuitively, it should be possible to find an effective section  $s$  avoiding a finite set of points if  $\bar{\mathcal{M}}_{a,b}$  has enough effective sections. Lemma 3.7 makes this intuition precise.

**Lemma 3.6.** *Suppose  $\bar{\mathcal{L}}$  is a relatively semipositive hermitian line bundle on  $\mathcal{X}$  such that  $\deg(\mathcal{L}) > 0$  and  $\bar{\mathcal{L}}^2 < 0$ . Then there exists an ample hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  such that for all  $a, b \in \mathbb{N}$  satisfying*

$$\bar{\mathcal{M}}_{a,b}^2 = \left( \bar{\mathcal{M}}^{\otimes a} \otimes \bar{\mathcal{L}}^{\otimes b} \right)^2 > 0,$$

*the lattice  $H^0(\mathcal{X}, \bar{\mathcal{M}}_{a,b}^{\otimes n})$  has a basis consisting of strictly effective global sections for some  $n = n(a, b) \gg 0$ .*

*Proof.* Let  $\bar{\mathcal{M}}$  be an ample hermitian line bundle on  $\mathcal{X}$ . We will scale  $\bar{\mathcal{M}}$  by  $\alpha \in \mathbb{Q}_{\geq 0}$  such that the lemma holds for  $\bar{\mathcal{M}}(\alpha)$ . By [22, Theorem 1.5], it suffices to show that  $\bar{\mathcal{M}}_{a,b}(\alpha)^2 > 0$  implies that  $\bar{\mathcal{M}}_{a,b}(\alpha)$  is horizontally positive, since relative semipositivity is automatic.

Let  $p_\alpha(t)$  denote the polynomial

$$p_\alpha(t) = \bar{\mathcal{L}}^2 + 2(\bar{\mathcal{L}} \cdot \bar{\mathcal{M}}(\alpha))t + \bar{\mathcal{M}}(\alpha)^2 t^2.$$

As in the proof of Lemma 3.4, there is some positive real number  $t_0 = t_0(\alpha)$  such that  $p_\alpha(t_0) = 0$  and  $p_\alpha(t) > 0$  for all  $t > t_0$ .

Now let  $m_{\mathcal{L}} = -\inf_D h_{\bar{\mathcal{L}}}(D)$  and  $m_{\mathcal{M}} = \inf_D h_{\bar{\mathcal{M}}}(D)$ , where we take the infima over all irreducible divisors  $D$  on  $X$ . If  $m_{\mathcal{L}} \geq 0$ , then we can take  $\alpha = 0$ , so we may assume that  $m_{\mathcal{L}} < 0$ .

Let  $D$  be some irreducible divisor on  $X$ . We will construct  $\alpha$  such that

$$(17) \quad h_{\bar{\mathcal{M}}_{a,b}(\alpha)}(D) \geq 0$$

whenever  $a/b > t_0(\alpha)$ , which will prove the lemma.

Because of

$$\begin{aligned} h_{\bar{\mathcal{M}}_{a,b}(\alpha)}(D) &= ah_{\bar{\mathcal{M}}(\alpha)}(D) + bh_{\bar{\mathcal{L}}}(D) \\ &\geq am_{\mathcal{M}} + a\alpha - bm_{\mathcal{L}}, \end{aligned}$$

we need a nonnegative  $\alpha$  such that  $\frac{a}{b} \geq t_0(\alpha) \Rightarrow \frac{a}{b} \geq \frac{m_{\mathcal{L}}}{\alpha + m_{\mathcal{M}}}$ , so  $\alpha$  must satisfy

$$t_0(\alpha) \geq \frac{m_{\mathcal{L}}}{\alpha + m_{\mathcal{M}}}.$$

Hence (17) is easily seen to follow from

$$(18) \quad (\alpha + m_{\mathcal{M}})r(\alpha) \geq m_{\mathcal{L}} \left( \bar{\mathcal{M}}^2 + 2\alpha \deg(\mathcal{M}) \right) + (\alpha + m_{\mathcal{M}}) \left( (\bar{\mathcal{M}} \cdot \bar{\mathcal{L}}) + \alpha \deg(\mathcal{L}) \right),$$

where

$$r(\alpha) = \sqrt{\alpha^2 \deg(\mathcal{L})^2 + 2\alpha \deg(\mathcal{M}) \left( (\bar{\mathcal{M}} \cdot \bar{\mathcal{L}}) - \bar{\mathcal{L}}^2 \right) + (\bar{\mathcal{M}} \cdot \bar{\mathcal{L}})^2 - \bar{\mathcal{L}}^2 \bar{\mathcal{M}}^2}.$$

Note that  $r(\alpha)$  is real since  $p_\alpha$  always has real roots. Hence the left hand side of (18) is always nonnegative. If the right hand side is negative for some  $\alpha$ , then (18) holds and we are done, so we may assume that the right hand side is also nonnegative.

We find that (18) holds if and only if  $w(\alpha) \geq 0$ , where  $w$  is a cubic polynomial in  $\alpha$ , obtained by subtracting the square of the right hand side of (18) from the square of the left hand side. The leading coefficient of  $w$  is

$$-2 \deg(\mathcal{M})(\bar{\mathcal{L}}^2 + 2m_{\mathcal{L}} \deg(\mathcal{L}))$$

which is positive by our assumptions on  $\bar{\mathcal{L}}$ . Hence (18) holds for  $\alpha \gg 0$ .  $\square$

**Lemma 3.7.** *Let  $\bar{\mathcal{M}}$  be an ample hermitian line bundle on  $\mathcal{X}$  and let  $P_1, \dots, P_r$  be closed points on  $\mathcal{X}$ . Then for some  $n \gg 0$  there exists a strictly effective global section  $s$  of  $\mathcal{M}^{\otimes n}$  such that  $s(P_i) \neq 0$  for  $i = 1, \dots, r$ .*

*Proof.* Without loss of generality we assume that  $\bar{\mathcal{M}}$  is very ample. Since  $\bar{\mathcal{M}}$  is very ample as a hermitian line bundle, the lattice  $H^0(X, \mathcal{M})$  is spanned by strictly effective sections  $s_1, \dots, s_m$ . Moreover, since  $\mathcal{M}$  is also very ample in the geometric setting, it is globally generated, i.e., for each point  $P_i$  there exists at least one section  $s_j$  such that  $s_j(P_i) \neq 0$ . Let  $\{s_1, \dots, s_k\}$  be a minimal set of such sections. Then for every even  $n$  the global section

$s' = \sum_{j=1}^k s_j^n$  of  $\mathcal{M}^{\otimes n}$  satisfies  $s'(P_i) \neq 0$  for all  $i = 1, \dots, r$ . But now for some even  $n \gg 0$  the global section  $s' = \sum_{j=1}^k s_j^n$  also satisfies the condition

$$\|s'\|_{\text{sup}} \leq k \max_j \|s_j\|^n < 1.$$

□

#### 4. HEIGHTS AND INTERSECTIONS

We keep the notation of the previous section. If  $F$  is a finite extension of  $K$ , then we let  $\varphi_F = \text{pr}_1 \circ \pi_F : \mathcal{X}^F \rightarrow \mathcal{X}$ , where  $\pi_F : \mathcal{X}^F \rightarrow \mathcal{X} \times \mathcal{O}_F$  denotes the minimal desingularization. For the definition of semistable arithmetic surfaces we refer to [14]; in particular, we do not require a semistable arithmetic surface to be minimal.

**Lemma 4.1.** (*Liu, [14]*) *There exists a finite extension  $F_0/K$  such that  $\mathcal{X}^F$  is semistable for every finite extension  $F/F_0$ .*

*Definition 4.2.* Let  $F_0/K$  be as in Lemma 4.1. We denote the smooth locus of  $\mathcal{X}^{F_0}$  by  $\mathcal{X}_{\text{sm}}^{F_0}$  and we denote the exceptional locus of  $\varphi_{F_0}$  by  $\text{Exc}(\varphi_{F_0})$ . With this notation we define

$$T(\mathcal{X}) = \varphi_{F_0}(\mathcal{X}^{F_0} \setminus \mathcal{X}_{\text{sm}}^{F_0} \cup \text{Exc}(\varphi_{F_0})).$$

*Remark 4.3.* If  $\mathcal{X}$  is semistable, then we have  $T(\mathcal{X}) = \mathcal{X} \setminus \mathcal{X}_{\text{sm}}$ .

**Lemma 4.4.** *Every irreducible divisor  $E$  on  $X$  such that  $\text{supp}(E_{\mathcal{X}}) \cap T(\mathcal{X}) = \emptyset$  satisfies  $\varphi_F^* E_{\mathcal{X}} = E_{\mathcal{X}^F}$  for every finite extension  $F/F_0$ .*

*Proof.* Let  $E$  be an irreducible divisor on  $X$  whose closure  $E_{\mathcal{X}}$  does not contain an element of  $T(\mathcal{X})$  in its support and let  $F/K$  be a finite extension containing  $F_0$ . Note that if  $\varphi^* E_{\mathcal{X}} \neq E_{\mathcal{X}^F}$ , then there is an irreducible component  $\Gamma \subset \text{Exc}(\varphi_F)$  such that  $\varphi_F(\Gamma) \in \text{supp}(E_{\mathcal{X}})$ .

But this means that either  $\Gamma \subset \text{Exc}(\varphi_{F_0})$ , implying that  $\varphi_F(\Gamma) \in T(\mathcal{X})$ , or  $\Gamma$  is contracted to a point by the desingularization morphism  $\mathcal{X}_F \rightarrow \mathcal{X}_{F_0} \times \mathcal{O}_F$ . In this case  $\Gamma$  maps to a singular point of  $\mathcal{X}_{F_0} \times \mathcal{O}_F$ , whence  $\varphi_F(\Gamma) \in T(\mathcal{X})$ . □

Let  $h_{\text{NT}}$  denote the Néron-Tate height on the Jacobian  $J$  of  $X$  with respect to the symmetrized theta divisor  $\Theta + [-1]^* \Theta$ . For each divisor  $D \in \text{Div}_{\mathbb{Q}}(X)$  of degree one, let  $j_D : X \hookrightarrow J$  be the embedding which maps a point  $Q \in X$  to the class of  $Q - D$ .

*Definition 4.5.* Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  have degree one. For every non-archimedean place  $v$  of  $K$  we define  $D_v = D \times K_v^{\text{nr}}$ , where  $K_v^{\text{nr}}$  is the maximal unramified extension of the completion of  $K$  at  $v$  and we fix the proper regular model  $\mathcal{X} \times \mathcal{O}_v^{\text{nr}}$  of  $X$  over the ring  $\mathcal{O}_v^{\text{nr}}$  of integers of  $K_v^{\text{nr}}$ . With these choices, we define vertical divisors

$$V_D = \sum_v V_{D_v}$$

and

$$U_D = \sum_v U_{D_v}$$

on  $\mathcal{X}$ , where both sums are over all non-archimedean places of  $K$ . Moreover, we define a hermitian line bundle  $\overline{\mathcal{L}}_D$  on  $\mathcal{X}$  by

$$\overline{\mathcal{L}}_D = \overline{\omega} \otimes \overline{\mathcal{O}}(2D_{\mathcal{X}}) \otimes \mathcal{O}(U_D)^{-1} \otimes \overline{\mathcal{O}}(-a\mathcal{X}_{\infty}),$$

where  $a = \bar{\mathcal{O}}(D_{\mathcal{X}})^2 - \mathcal{O}(V_D)^2 \in \mathbb{R}$ . Here  $\bar{\mathcal{O}}(D_{\mathcal{X}})^2$  is the self-intersection of  $\mathcal{O}(D_{\mathcal{X}})$ , equipped with the Arakelov metric, and  $\bar{\mathcal{O}}(-a\mathcal{X}_{\infty}) = (\mathcal{O}_{\mathcal{X}}, |\cdot|e^a)$ . Finally, we set

$$\beta_D = \sum_v \beta_{D_v}.$$

Unless otherwise stated, all metrics will be Arakelov metrics (so that the Arakelov adjunction formula holds, cf. [12, §IV.5]), except for vertical line bundles, which are equipped with the trivial metric.

Now we prove Theorem 1.1, stating that the height with respect to  $\bar{\mathcal{L}}_D$  is closely related to the Néron-Tate height on the Jacobian of  $X$ .

*Proof of Theorem 1.1.* Let  $E$  be an irreducible divisor on  $X$  such that  $\text{supp}(E_{\mathcal{X}}) \cap T(\mathcal{X}) = \emptyset$ . Let  $F_0$  be as in Lemma 4.1 and let  $F/K$  be a finite extension containing  $F_0$  such that  $E$  has pointwise  $F$ -rational support. Let  $e = \deg(E)$  and  $E = \sum_{j=1}^e (P_j)$ , where  $P_j \in X(F)$ . By Lemma 4.4 we have

$$(19) \quad \varphi_F^* E_{\mathcal{X}} = E_{\mathcal{X}^F} \quad \text{and} \quad \varphi_F^* D_{\mathcal{X}} = D_{\mathcal{X}^F}.$$

Hence we get, using [12, Theorem III.4.5],

$$(20) \quad \Phi_{\mathcal{X}^F}(E - eD) = \varphi_F^*(\Phi_{\mathcal{X}}(E - eD)),$$

where, if  $Z \in \text{Div}_{\mathbb{Q}}(X)$  has degree 0,  $\Phi_{\mathcal{X}}(Z) \in \text{Div}_{\mathbb{Q}}(\mathcal{X})$  is a vertical divisor such that  $Z_{\mathcal{X}} + \Phi_{\mathcal{X}}(Z)$  has trivial intersection multiplicity with all vertical divisors on  $\mathcal{X}$ . In our notation, we have  $\Phi_{\mathcal{X}}(Z) = \Phi_{\mathcal{X} \times_{\mathcal{O}_v} \mathcal{O}_v}(Z_v) = \sum_v V_{Z_v}$ , see (3). Expanding the left hand side of (20), we find

$$\Phi_{\mathcal{X}^F}(E - eD) = \sum_{j=1}^e \Phi_{\mathcal{X}^F}(P_j - D)$$

and Proposition 2.6 implies that

$$(21) \quad (\bar{\mathcal{O}}(E_{\mathcal{X}}) \cdot \mathcal{O}(U_D)) = e\mathcal{O}(V_D)^2 - \sum_{j=1}^e \mathcal{O}(\Phi_{\mathcal{X}^F}(P_j - D))^2.$$

This allows us to compare  $h_{\bar{\mathcal{L}}_D}$  to  $h_{\text{NT}}$ . We will use the Hodge Index Theorem on arithmetic surfaces due to Faltings and Hriljac (see for instance [12, §III.5]) which implies that if  $P \in J(K)$ , then we have

$$(22) \quad h_{\text{NT}}(P)[K : \mathbb{Q}] = -(\bar{\mathcal{O}}(Z_{\mathcal{X}}) \otimes \mathcal{O}(\Phi_{\mathcal{X}}(Z)) \cdot \bar{\mathcal{O}}(Z_{\mathcal{X}})) = -\bar{\mathcal{O}}(Z_{\mathcal{X}})^2 + \mathcal{O}(\Phi_{\mathcal{X}}(Z))^2$$

for every divisor  $Z$  of degree zero on  $X$  such that  $Z$  represents  $P$ .

Setting  $a' = -\overline{\mathcal{O}}(D_{\mathcal{X}})^2$ , we get

$$\begin{aligned}
\sum_{j=1}^e h_{\text{NT}}(j_D(P_j))[K : \mathbb{Q}] &= \sum_{j=1}^e \left( -\overline{\mathcal{O}}(P_{j,\mathcal{X}^F} - D_{\mathcal{X}^F})^2 + \mathcal{O}(\Phi_{\mathcal{X}^F}(P_j - D))^2 \right) \\
&= \sum_{j=1}^e \left( -\overline{\mathcal{O}}(P_{j,\mathcal{X}^F})^2 + 2(\overline{\mathcal{O}}(P_{j,\mathcal{X}^F}) \cdot \overline{\mathcal{O}}(D_{\mathcal{X}^F})) + \mathcal{O}(\Phi_{\mathcal{X}^F}(P_j - D))^2 \right) - ea' \\
&= \sum_{j=1}^e \left( (\overline{\mathcal{O}}(P_{j,\mathcal{X}^F}) \cdot \overline{\omega}_{\mathcal{X}^F} \otimes \overline{\mathcal{O}}(2D_{\mathcal{X}^F})) + \mathcal{O}(\Phi_{\mathcal{X}^F}(P_j - D))^2 \right) - ea' \\
&= (\overline{\mathcal{O}}(E_{\mathcal{X}^F}) \cdot \overline{\omega}_{\mathcal{X}^F} \otimes \overline{\mathcal{O}}(2D_{\mathcal{X}^F})) + \sum_{j=1}^e \mathcal{O}(\Phi_{\mathcal{X}^F}(P_j - D))^2 - ea' \\
&= (\overline{\mathcal{O}}(E_{\mathcal{X}}) \cdot \overline{\omega} \otimes \overline{\mathcal{O}}(2D_{\mathcal{X}})) - (\overline{\mathcal{O}}(E_{\mathcal{X}}) \cdot \mathcal{O}(U_D)) - ea' + e\mathcal{O}(V_D)^2 \\
&= (\overline{\mathcal{O}}(E_{\mathcal{X}}) \cdot \overline{\mathcal{L}}_D) \\
&= e[K : \mathbb{Q}]h_{\overline{\mathcal{L}}_D}(E).
\end{aligned}$$

Here the first equality holds by (22), the third equality holds because of the Arakelov adjunction formula (see [12, §IV.5]) and the fifth equality holds because of (19), (21) and because, by assumption,  $E_{\mathcal{X}^i}$  does not intersect any vertical divisors contracted by  $\varphi_F$ .

The first assertion of the proposition is now immediate since the Néron-Tate height only takes nonnegative values. The second assertion follows if we put  $E = (P)$ , where  $P \in X(K)$ .  $\square$

*Remark 4.6.* If  $X$  is a smooth projective geometrically irreducible curve defined over an archimedean local field and  $E_1, E_2 \in \text{Div}(X)$  have disjoint support, then we set  $[E_1, E_2] = (E_1, E_2)_a$ , where the latter denotes the admissible pairing on  $X$ , see [23, §4.5].

Now suppose that  $X$  is defined over a number field  $K$ . We can use (1) and Corollary 2.2 to define a pairing on divisors  $E_1, E_2$  on  $X$  with disjoint support as

$$[E_1, E_2] = \sum_v [E_{1,v}, E_{2,v}]_v,$$

where  $E_{i,v} = E_i \times_K K_v$  for archimedean  $v$  and the sum is over all places of  $K$ . This global pairing has the following properties, which may be of independent interest:

- (i)  $[\cdot, \cdot]$  is bilinear and symmetric.
- (ii)  $[E_1, \text{div}(f)] = 0$  for any  $f \in K(X)^*$ . Hence  $[\cdot, \cdot]$  induces a well-defined pairing on divisor classes.
- (iii) If  $\deg(E_1) = \deg(E_2) = 0$ , then we have  $[E_1, E_2] = -(E_1, E_2)_{\text{NT}}$ , where the latter is the Néron-Tate height pairing.
- (iv) If  $E_1$  and  $E_2$  are canonical divisors on  $X$ , then we have  $[E_1, E_2] = \overline{\omega}^2$ .

## 5. PROOFS OF PROPOSITION 1.2, THEOREM 1.3 AND THEOREM 1.4

We finally get to our original problem, namely the derivation of lower bounds on  $\overline{\omega}^2$ . We first prove Proposition 1.2. If  $\overline{\mathcal{M}}$  is a hermitian line bundle on  $\mathcal{X}$  and  $a, b$  are positive integers, then we set

$$\overline{\mathcal{M}}_{a,b} = \overline{\mathcal{M}}^{\otimes a} \otimes \overline{\mathcal{L}}_D^{\otimes b}.$$



We want to use Lemma 3.4 to prove Proposition 1.2, so we need to show that under the hypothesis  $\bar{\mathcal{L}}_D^{-2} < 0$  there is some hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  with positive self-intersection such that  $\bar{\mathcal{M}}_{a,b}^2 > 0$  implies  $(\bar{\mathcal{L}}_D \cdot \bar{\mathcal{M}}_{a,b}) \geq 0$ .

**Proposition 5.1.** *Suppose that  $\bar{\mathcal{L}}_D^{-2} < 0$  and that  $D_{\mathcal{X}} \cap T(\mathcal{X}) = \emptyset$ . Then there exists an ample hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  such that for any positive integers  $a, b$  the following condition is satisfied: If  $\bar{\mathcal{M}}_{a,b}^2 > 0$ , then there exists a positive integer  $n(a, b)$  such that  $\bar{\mathcal{M}}_{a,b}^{\otimes n(a,b)}$  has an effective section  $s$  satisfying  $h_{\bar{\mathcal{L}}_D}(\text{div}(s)^{\text{hor}}) \geq 0$ .*

*Proof.* It follows from Lemma 3.6 that there is an ample hermitian line bundle  $\bar{\mathcal{M}}$  on  $\mathcal{X}$  such that  $\bar{\mathcal{M}}_{a,b}^2 > 0$  implies that  $H^0(\mathcal{X}, \bar{\mathcal{M}}_{a,b}^{\otimes n})$  has a basis consisting of strictly effective sections for  $n$  large enough. By [22, Theorem 1.3],  $\bar{\mathcal{M}}_{a,b}$  is ample.

Lemma 3.7 implies that there is a multiple  $n(a, b)$  of  $n$  and an effective section  $s$  of  $\bar{\mathcal{M}}_{a,b}^{\otimes n(a,b)}$  such that  $\text{div}(s)^{\text{hor}}$  does not intersect the finite set  $T(\mathcal{X}) \subset \mathcal{X}$ . Using Theorem 1.1 we conclude  $h_{\bar{\mathcal{L}}_D}(\text{div}(s)^{\text{hor}}) \geq 0$ .  $\square$

Now we can complete the proof of Proposition 1.2.

*Proof of Proposition 1.2.* Suppose that  $\bar{\mathcal{L}}_D^{-2} < 0$ . Let  $\bar{\mathcal{M}}$  be as in Proposition 5.1 and let  $a, b$  be positive integers such that  $\bar{\mathcal{M}}_{a,b}^2 > 0$ . It follows from Lemma 3.5 and Proposition 5.1 that we have

$$(\bar{\mathcal{L}}_D \cdot \bar{\mathcal{M}}_{a,b}^{\otimes n(a,b)}) \geq 0$$

and thus

$$(\bar{\mathcal{L}}_D \cdot \bar{\mathcal{M}}_{a,b}) \geq 0.$$

But by Lemma 3.4 this leads to a contradiction.  $\square$

Next we prove Theorem 1.3. It follows from [5, Lemma A.1] that there exists a  $\mathbb{Q}$ -divisor  $D \in \text{Div}_{\mathbb{Q}}(X)$  such that  $D_{\mathcal{X}} \cap T(\mathcal{X}) = \emptyset$  and such that  $(2g - 2)D$  is a canonical  $\mathbb{Q}$ -divisor on  $X$ . Moreover, it is shown in [8] that

$$(23) \quad \mathcal{K} = (2g - 2)(D_{\mathcal{X}} + V_D) \in \text{Div}_{\mathbb{Q}}(\mathcal{X})$$

is a canonical  $\mathbb{Q}$ -divisor on  $\mathcal{X}$ . By the latter we mean a  $\mathbb{Q}$ -divisor such that  $\mathcal{O}(\mathcal{K}) = \omega$ .

*Proof of Theorem 1.3.* From (23) we get  $D_{\mathcal{X}} = \frac{1}{2g-2}\mathcal{K} - V_D$ . We can use this to rewrite  $\bar{\mathcal{L}}_D$  (cf. Definition 4.5) as

$$\bar{\mathcal{L}}_D = \bar{\omega}^{\otimes \frac{g}{g-1}} \otimes \mathcal{O}(-2V_D - U_D) - 4ga,$$

where

$$a = \bar{\mathcal{O}}(D_{\mathcal{X}})^2 - \mathcal{O}(V_D)^2 = \frac{1}{4(g-1)^2}\bar{\omega}^2 - \frac{1}{g-1}(\bar{\omega} \cdot \mathcal{O}(V_D)).$$

Hence we have

$$\bar{\mathcal{L}}_D^{-2} = \frac{1}{g-1} (g\bar{\omega}^2 + (g-1)\mathcal{O}(2V_D + U_D)^2 - 2g(\bar{\omega} \cdot \mathcal{O}(U_D))).$$

Since  $\bar{\mathcal{L}}_D^{-2} \geq 0$  by Proposition 1.2, Theorem 1.3 follows.  $\square$

*Remark 5.2.* We have developed the theory of  $\overline{\mathcal{L}}_D$  for rather general degree one divisors  $D \in \text{Div}_{\mathbb{Q}}(X)$ . The main reason why we choose to work with  $D$  as in Theorem 1.3 is that  $D_{\mathcal{X}}$  has an obvious relation with  $\overline{\omega}$ . But there are other promising choices for  $D$ ; for instance, we could take  $D = \frac{1}{g^3-g}W$ , where  $W$  is the divisor of Weierstrass points on  $X$ . This was suggested by Ariyan Javanpeykar. In fact, it is easy to see that the divisor  $\mathcal{V}$  used in [9, Lemma 5.1] to extend  $W$  to a divisor on  $\mathcal{X}$  with good properties is a valid choice for  $V_W$ .

Now we derive Theorem 1.3 from Theorem 1.4.

*Proof of Theorem 1.4.* Let  $D \in \text{Div}_{\mathbb{Q}}(X)$  have degree one. We know that  $\beta = \beta_D$  does not depend on the choice of  $D$  because of Lemma 2.14. In order to apply Theorem 1.3, we first need to show that  $\overline{\mathcal{L}}_D$  is relatively semipositive. Let  $\Gamma$  be an irreducible component of a special fiber of  $\mathcal{X}$ . By definition of  $\overline{\mathcal{L}}_D$ , we have

$$(\overline{\mathcal{L}}_D \cdot \mathcal{O}(\Gamma)) = (\mathcal{K} \cdot \Gamma) + 2(D_{\mathcal{X}} \cdot \Gamma) - (U_D \cdot \Gamma)$$

up to a positive rational constant, where  $\mathcal{K}$  is a canonical divisor on  $\mathcal{X}$ . Hence relative semipositivity of  $\overline{\mathcal{L}}_D$  follows from part (b) of Lemma 2.11.

Since we know that the assumptions of Theorem 1.3 are satisfied, it suffices to show that  $(\mathcal{K} \cdot U_D) \geq 0$ . But this is an immediate consequence of Lemma 2.12.  $\square$

*Remark 5.3.* One of the main original motivations to look at lower bounds for  $\overline{\omega}^2$  was a conjecture of Bogomolov. Building on earlier work of Zhang [23], the conjecture was finally proved by Ullmo [21], who proved the positivity of the admissible self-intersection  $\omega_a^2$  of  $\omega$  for curves over number fields. For curves over function fields of characteristic 0, the conjecture was reduced by Zhang [24] to a conjecture about invariants of polarized metrized graphed and the latter was proved by Cinkir [6]. However, Cinkir actually proved an *effective* version of the Bogomolov conjecture.

Such an effective version can also be conjectured over number fields, but in this situation it has not been proved yet (it would follow from a proof of the arithmetic standard conjectures of Gillet-Soulé, see [24, §1.4]). Using [23], it suffices to find an effectively computable nontrivial lower bound for  $\omega_a^2$ . If  $\mathcal{X}$  is semistable and minimal, then we have

$$\overline{\omega}^2 = \omega_a^2 - \sum_v \varepsilon_v(X),$$

where  $\varepsilon_v(X) \geq 0$  is Zhang's admissible constant associated to  $X \times K_v$  (see Remark 2.16) and the sum is over all non-archimedean places of  $K$ . Hence it suffices to find an effectively computable lower bound  $b$  for  $\overline{\omega}^2$  such that  $\sum_v \varepsilon_v(X) < b$ . Therefore our work provides a possible approach to the effective Bogomolov Conjecture, but unfortunately we already have  $\beta \leq \sum_v \varepsilon_v$  for  $g = 2$  by Remark 2.17.

## 6. APPLICATIONS

Now we apply our results to compute lower bounds on the self-intersection of the relative dualizing sheaf for certain families of curves.

**6.1. Modular curves.** Let  $N = N'QR$  be a squarefree integer such that  $Q, R \geq 4$  and  $\gcd(Q, R) = 1$ . Consider the modular curve  $X_1(N)$  over the cyclotomic field  $\mathbb{Q}[\zeta_N]$  and its minimal regular model  $\mathcal{X} = \mathcal{X}_1(N)/\mathbb{Z}[\zeta_N]$ . Then  $\mathcal{X}$  has semistable and reduced fibers. More

precisely, the special fibers  $\mathcal{X}_{\mathfrak{p}}$  are smooth if  $\mathfrak{p} \nmid N$ . If  $\mathfrak{p} \mid N$  with residue characteristic  $p$ , then the special fiber  $\mathcal{X}_{\mathfrak{p}}$  consists of two isomorphic curves intersecting in

$$s_{\mathfrak{p}} = \frac{p-1}{24} \frac{\varphi(N/p)N}{p} \prod_{q \mid N/p} \left(1 + \frac{1}{q}\right)$$

points, all of which are rational over the residue field at  $\mathfrak{p}$ , see [15, Proposition 7.3]. The arithmetic genera of these components are given by

$$q_{\mathfrak{p}} = \frac{1}{2}(g_N - s_{\mathfrak{p}} + 1),$$

where

$$g_N = 1 + \frac{1}{24} \varphi(N)N \prod_{p \mid N} \left(1 + \frac{1}{p}\right) - \frac{1}{4} \sum_{d \mid N} \varphi(d) \varphi(N/d)$$

is the genus of  $X_1(N)$ .

We can use Example 2.15 to compute an asymptotic lower bound for  $\bar{\omega}^2$  quite easily:

**Proposition 6.1.** *The arithmetic self-intersection  $\bar{\omega}^2$  of the relative dualizing sheaf on  $\mathcal{X}(N)$  satisfies*

$$\bar{\omega}^2 \geq \frac{1}{2} \varphi(N) \log N + o(1),$$

*Proof.* Let  $n_{\mathfrak{p}} = \log \#k(\mathfrak{p})$ . Then we have  $\sum_{\mathfrak{p} \mid p} n_{\mathfrak{p}} = \varphi(N/p) \log(p)$  and hence, by Example 2.15,

$$\begin{aligned} \beta &= \sum_{\mathfrak{p} \mid N} \frac{n_{\mathfrak{p}}}{2s_{\mathfrak{p}}} (s_{\mathfrak{p}} + 2q_{\mathfrak{p}} - 2) \\ &= \sum_{\mathfrak{p} \mid N} \frac{n_{\mathfrak{p}}}{2s_{\mathfrak{p}}} (g_N - 1) \\ &= \frac{g_N - 1}{2} \sum_{p \mid N} \sum_{\mathfrak{p} \mid p} \frac{n_{\mathfrak{p}}}{s_{\mathfrak{p}}} \\ &= 12(g_N - 1) \frac{\varphi(N)}{\prod_{p \mid N} p^2 - 1} \sum_{p \mid N} \frac{p+1}{p-1} \log p \\ &= \frac{1}{2} \varphi(N) \log N + o(1), \end{aligned}$$

since  $\frac{24(g_N-1)}{\prod_{p \mid N} p^2 - 1} = 1 + o(1)$ . □

*Remark 6.2.* In [15, Theorem 7.7], Mayer obtains the asymptotic formula

$$\bar{\omega}^2 = 3g_N \log(N) + o(g_N \log(N)).$$

Our lower bound is much smaller than this asymptotic value.

**6.2. Fermat curves of prime exponent.** In Section 5 we derived a nontrivial lower bound  $\beta_D$  on  $\bar{\omega}^2$  for minimal arithmetic surfaces with simple multiplicities. In the present subsection we compute lower bounds on  $\bar{\omega}^2$  in a situation where Theorem 1.4 is not applicable, namely for minimal regular models of Fermat curves of prime exponent. Along the way, we construct  $U_D$  and show that  $\overline{\mathcal{L}}_D$  is relatively semipositive, where  $D = (S_x)$  for a certain rational point  $S_x$  such that  $(2g - 2)D$  is a canonical  $\mathbb{Q}$ -divisor on  $\mathcal{X}$ .

We start with a brief review of the notation from [8]. Let  $p > 3$  be a prime number, let

$$F_p : X^p + Y^p = Z^p$$

denote the Fermat curve with exponent  $p$  over  $K = \mathbb{Q}(\zeta_p)$ , where  $\zeta_p$  is a primitive  $p$ -th root of unity. Let  $\mathcal{F}_p^{\min}$  denote the minimal regular model of  $F_p$  over  $\mathbb{Z}[\zeta_p]$  as computed by McCallum cf. [16]. We denote the components of the only non-reduced special fiber of  $\mathcal{F}_p^{\min}$  by  $L_i, i \in I$ , where

$$I = \{x, y, z, \alpha_1, \alpha_{1,1}, \dots, \alpha_{1,p}, \alpha_2, \alpha_{2,1}, \dots, \alpha_{2,p}, \dots, \alpha_r, \alpha_{r,1}, \dots, \alpha_{r,p}, \beta_1, \dots, \beta_s\}$$

and  $2r + s = p - 3$ , see [8]. In Figure 1 the configuration of the only reducible special fiber (occurring at the unique prime above  $p$ ) of a certain non-minimal model  $\mathcal{F}_p$  of  $F_p$  is shown. It has the property that contracting the unique exceptional component  $L$  on  $\mathcal{F}_p$  yields  $\mathcal{F}_p^{\min}$ . For every component  $L_i$ , we also list the pair  $(m_i, L_i^2)$ , where  $m_i$  is the multiplicity of  $L_i$  in  $\mathcal{F}_p$ . All components have genus 0 and the only component with self-intersection number -1 is  $L$ . See [16, 8] for further details.

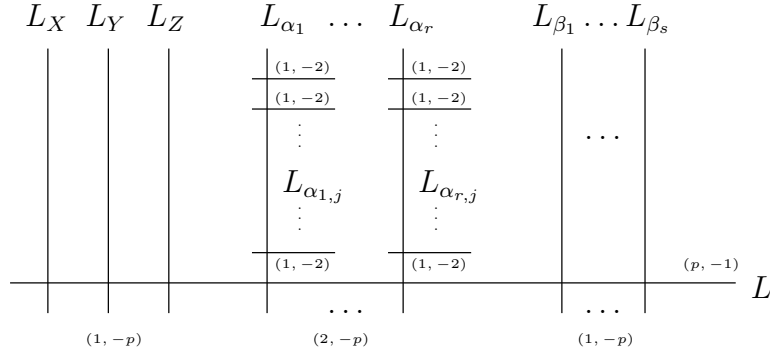


FIGURE 1. The configuration of the only reducible special fiber of  $\mathcal{F}_p$ .

Note that  $L_{\alpha_i}$  has multiplicity two for all  $i$ . Therefore Theorem 1.4 does not apply in general and we have to show relative semipositivity of  $\overline{\mathcal{L}}_D$  and nontriviality of the bound from Theorem 1.3 directly for a suitable  $\mathbb{Q}$ -divisor  $D$ . Since there is only one place of bad reduction, we will omit it from the notation for the sake of simplicity.

It is shown by Curilla and the first author in [8] that a canonical divisor on  $F_p$  is given by  $(2g - 2)D$ , where  $D = S_x$  for a certain  $K$ -rational point  $S_x$  on  $F_p$  whose Zariski closure  $S_x := S_{x, \mathcal{F}_p^{\min}}$  in  $\mathcal{F}_p^{\min}$  intersects only  $L_x$ . In order to compute a lower bound on  $\bar{\omega}^2$  using the results of the previous sections, we will find  $U_D$ . This means that we first have to compute the divisor  $V_i$  for each  $i \in I$ .

**Lemma 6.3.** *We can take*

- $V_i = \frac{1}{p}L_i$ , if  $i \in \{x, y, z, \beta_j\}$ ,
- $V_{\alpha_i} = \frac{1}{p}L_{\alpha_i} + \frac{1}{2p} \sum_{j=1}^p L_{\alpha_{ij}}$ ,
- $V_{\alpha_{i,j}} = -\frac{1}{p}L_{\alpha_i} + \left(\frac{1}{2} - \frac{1}{2p}\right) L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}}$ .

*Proof.* Recall that the divisor  $V_i$  must satisfy

$$(24) \quad (V_i \cdot L_j) = a'_j - \delta_{ij}$$

for all  $j \in I$ , where  $a'_j = \frac{1}{2g-2}(-L_j^2 + 2p_a(L_j) - 2)$  and  $\delta_{ij}$  is the Kronecker delta function on  $I$ . Since all components have genus zero and self-intersection  $1-p$  except for the components  $L_{\alpha_{ij}}$ , which have genus zero and self-intersection  $-2$ , we get  $a'_j = 0$  for the components  $L_{\alpha_{ij}}$  and  $a'_j = \frac{1}{p}$  for all other components. Checking the validity of the Lemma reduces to checking (24) for each  $i \in I$  which is a simple computation that we leave to the reader. Note that for  $V_x$  this was essentially shown in [8, Proposition 8.3].  $\square$

**Corollary 6.4.** *A canonical  $\mathbb{Q}$ -divisor on  $\mathcal{F}_p^{\min}$  is given by  $(2g-2)\mathcal{S}_x + \frac{1}{p}L_x$  and we have*

$$\begin{aligned} U_D &= \frac{1-p}{p^2}L_x + \frac{1+p}{p^2} \left( L_y + L_z + \sum_{j=1}^s L_{\beta_j} \right) + \frac{1+p/2}{p^2} \sum_{i=1}^r L_{\alpha_i} \\ &\quad + \frac{p^2/2 + p/2 - 3}{p^2} \sum_{i=1}^r \sum_{j=1}^p L_{\alpha_{ij}}. \end{aligned}$$

*Proof.* The first assertion follows immediately from (23), the fact that  $(2g-2)\mathcal{S}_x$  is a canonical divisor on  $F_p$  and Lemma 6.3.

For the second assertion, recall the definition of  $U_D$ :

$$U_D = \sum_{i \in I} (2(V_i \cdot V_x) - V_i^2) L_i$$

Next we compute

$$\begin{aligned} V_i^2 &= \left(\frac{1}{p}L_i\right)^2 = \frac{1}{p^2}(1-p) \text{ for } i \in \{x, y, z, \beta_j\} \\ (V_i \cdot V_x) &= \left(\frac{1}{p}L_i \cdot \frac{1}{p}L_x\right) = \frac{1}{p^2} \text{ for } i \in \{y, z, \beta_j\} \\ (V_{\alpha_i} \cdot V_x) &= \left(\frac{1}{p}L_{\alpha_i} + \frac{1}{2p}L_{\alpha_{ij}} \cdot \frac{1}{p}L_x\right) = \frac{1}{p^2} \\ (V_{\alpha_{ij}} \cdot V_x) &= \left(-\frac{1}{p}L_{\alpha_i} + \left(\frac{1}{2} - \frac{1}{2p}\right)L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}} \cdot \frac{1}{p}L_x\right) = -\frac{1}{p^2} \end{aligned}$$

$$\begin{aligned}
(V_{\alpha_i})^2 &= \left( \frac{1}{p}L_{\alpha_i} + \frac{1}{2p}L_{\alpha_{ij}} \right)^2 = \frac{1}{p^2}(1-p) + \frac{p}{p^2} - \frac{2p}{4p^2} = \frac{1}{p^2} - \frac{1}{2p} \\
(V_{\alpha_{ij}})^2 &= \left( -\frac{1}{p}L_{\alpha_i} + \left( \frac{1}{2} - \frac{1}{2p} \right) L_{\alpha_{ij}} - \frac{1}{2p} \sum_{k \neq j} L_{\alpha_{ik}} \right)^2 \\
&= \frac{1}{p^2}(1-p) - \frac{1}{2} \left( 1 - \frac{1}{p} \right)^2 - \frac{1}{2p^2}(p-1) - \frac{1}{p} \left( 1 - \frac{1}{p} \right) + \frac{p-1}{p^2} \\
&= \frac{1}{2p^2}(2-p-p^2)
\end{aligned}$$

A simple computation proves the corollary.  $\square$

**Lemma 6.5.** *The hermitian line bundle  $\overline{\mathcal{L}}_D$  is relatively semipositive.*

*Proof.* Recalling Definition 4.5 of  $\overline{\mathcal{L}}_D$  we see that we need to show

$$(25) \quad a_i + 2(\mathcal{S}_x \cdot L_i) - (U_D \cdot L_i) \geq 0$$

for all  $i \in I$ , where  $a_i = 0$  for  $i = \alpha_{ij}$  and  $a_i = p - 3$  for all other components.

As usual, we will distinguish between the different components  $L_i$  to prove (25). Throughout, we will use that  $s = p - 3 - 2r$  to eliminate  $s$ . We start with  $L_x$  and find, using Corollary 6.4, that

$$(U_D \cdot L_x) = \frac{1-p}{p^2}(1-p) + (s+2)\frac{1+p}{p^2} + r\frac{1+p/2}{p^2} = 2 - \frac{2}{p} - \frac{r}{p^2} - \frac{3r}{2p}$$

which gives

$$a_x + 2(\mathcal{S}_x \cdot L_x) - (U_D \cdot L_x) \geq p - 3 > 0.$$

Let  $i \in \{y, z, \beta_j\}$ . Then we get

$$(U_D \cdot L_i) = \frac{1-p}{p^2} + \frac{1+p}{p^2}(1-p) + (s+1)\frac{1+p}{p^2} + r\frac{1+p/2}{p^2} = -\frac{2}{p} - \frac{r}{p^2} - \frac{3r}{2p}$$

and thus

$$a_i + 2(\mathcal{S}_x \cdot L_i) - (U_D \cdot L_i) \geq p - 3 > 0.$$

It remains to consider the components  $L_{\alpha_i}$  and  $L_{\alpha_{ij}}$ . The computations are similar to the ones above, but tedious and hence are omitted. The upshot is that we get inequalities

$$a_{\alpha_i} + 2(\mathcal{S}_x \cdot L_{\alpha_i}) - (U_D \cdot L_{\alpha_i}) = p - 3 - \frac{1}{2}p + 1 + \frac{5}{p} + \frac{r}{p^2} + \frac{3r}{2p} \geq \frac{p}{2} - 2 > 0$$

and

$$a_{\alpha_{ij}} + 2(\mathcal{S}_x \cdot L_{\alpha_{ij}}) - (U_D \cdot L_{\alpha_{ij}}) = 1 + \frac{1}{2p} - \frac{7}{p^2} > 0.$$

This shows that (25) is satisfied for all components  $L_i$ , which proves the lemma.  $\square$

**Theorem 6.6.** *Let  $p > 3$  be a prime number and let  $\overline{\omega}^2$  denote the arithmetic self-intersection of the relative dualizing sheaf on  $\mathcal{F}_p^{\min}$ .*

(i) We have

$$\begin{aligned} \bar{\omega}^2 \geq & \frac{1}{4p^3(p-1)(p-2)} \left( (4+2r)p^6 - (32+10r)p^5 + (10+19r)p^4 + (124 \right. \\ & \left. - r - 25r^2)p^3 + (-56+52r+31r^2)p^2 + (156-328r+112r^2)p \right. \\ & \left. + 144 - 24r + 60r^2 \right) \log p \end{aligned}$$

(ii) We have

$$\bar{\omega}^2 \geq \frac{(4p^6 - 32p^5 + \frac{13}{2}p^4 + \frac{73}{2}p^3 - 52p^2 + 144)}{4p^3(p-1)(p-2)} \log p.$$

This lower bound is positive for all  $p > 7$ . Furthermore, if  $p = 5$ , then we have  $\bar{\omega}^2 \geq \frac{188}{125} \log 5$  and if  $p = 7$ , then we have  $\bar{\omega}^2 \geq \frac{37277}{6860} \log 7$ .

*Proof.* By Theorem 1.3 and Lemma 6.5 we know that

$$\bar{\omega}^2 \geq \beta_D = -\frac{g-1}{g} \mathcal{O}(2V_D + U_D)^2 + 2(\bar{\omega} \cdot \mathcal{O}(U_D)).$$

We compute the terms on the right hand side.

First note that

$$\begin{aligned} 2V_D + U_D = & \frac{1+p}{p^2} \left( L_x + L_y + L_z + \sum_{j=1}^s L_{\beta_j} \right) \\ & + \frac{1+p/2}{p^2} \sum_{i=1}^r L_{\alpha_i} + \frac{p^2/2 + p/2 - 3}{p^2} \sum_{i=1}^r \sum_{j=1}^p L_{\alpha_{ij}} \end{aligned}$$

Using this, it is not hard to verify that

$$\begin{aligned} (2V_D + U_D)^2 = & -\frac{pr}{2} - \frac{r}{2} + 1 + \frac{1}{p} \left( \frac{7}{4}r - 5 \right) + \frac{1}{p^2} \left( \frac{25}{4}r^2 - 5r - 1 \right) \\ (26) \quad & + \frac{1}{p^3} (17 - 30r + 11r^2) + \frac{1}{p^4} (12 - 2r + 5r^2). \end{aligned}$$

For the computation of  $(\mathcal{K} \cdot U_D)$ , where  $\mathcal{K}$  is a canonical divisor on  $\mathcal{F}^{\min}$ , we use the adjunction formula. Namely, if  $\theta_i$  denotes the multiplicity of  $L_i$  in  $U_D$ , then we have

$$(\mathcal{K} \cdot U_D) = \sum_{i \in I} \theta_i a_i$$

and hence

$$(27) \quad (\mathcal{K} \cdot U_D) = (p-3) \left( \frac{1-p}{p^2} + (s+2) \frac{1+p}{p^2} + r \frac{1+p/2}{p^2} \right).$$

A combination of (26) and (27) proves (i) after a little algebra.

For (ii) we use (i) and  $0 \leq r \leq \frac{1}{2}p - \frac{3}{2}$ . To derive the lower bounds for  $p = 5, 7$ , we use that  $r = 0$  if  $p = 5$  and  $r = 2$  if  $p = 7$ .  $\square$

The proof of Theorem 1.6 follows immediately from Theorem 6.6.

*Remark 6.7.* The upper bound computed by Curilla and the first author in [8] is of order  $\mathcal{O}(gp \log p)$ , i.e. of order  $\mathcal{O}(p^3 \log p)$ .

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