ON THE STABILITY OF POLYHARMONIC SPLINE RECONSTRUCTION

Armin Iske

Department of Mathematics, University of Hamburg, Germany
iske@math.uni-hamburg.de

ABSTRACT

This paper concerns the numerical stability of polyharmonic spline reconstruction from multivariate irregular samples. It is shown that the Lagrange basis functions of the polyharmonic spline reconstruction scheme are scale-invariant. Immediate consequences of this result on the conditioning of the reconstruction problem are first discussed, before a suitable preconditioner is developed.

Keywords— polyharmonic splines, irregular sampling.

1. INTRODUCTION

Irregular sampling requires reliable and robust methods from scattered data approximation. Polyharmonic splines, due to Duchon [1], are popular tools for multivariate Lagrange interpolation [5]. In this problem, a data vector

$$f\big|_X = (f(x_1), \ldots, f(x_n))^T \in \mathbb{R}^n$$

of function values, irregularly sampled from an unknown function $f : \mathbb{R}^d \to \mathbb{R}$ at a point set $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}^d$, $d \geq 1$, is assumed to be given. Interpolation from Lagrange data $f\big|_X$ requires the construction of a *suitable* interpolant $s : \mathbb{R}^d \to \mathbb{R}$ satisfying $s\big|_X = f\big|_X$, i.e.,

$$s(x_j) = f(x_j) \qquad \text{for all } 1 \leq j \leq n. \tag{1}$$

Polyharmonic spline interpolation solves (1), for $k \in \mathbb{N}$ satisfying $k > d/2$, in combination with the variational problem

$$|s|_{\mathbf{BL}^k(\mathbb{R}^d)} \leq |f|_{\mathbf{BL}^k(\mathbb{R}^d)} \qquad \text{with } s\big|_X = f\big|_X, \tag{2}$$

in the Beppo Levi space

$$\mathbf{BL}^k(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) : D^\alpha f \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| = k \right\},$$

being equipped with the semi-norm

$$|f|^2_{\mathbf{BL}^k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_{|\alpha|=k} \binom{k}{\alpha} (D^\alpha f)^2 \, dx \qquad \text{for } f \in \mathbf{BL}^k(\mathbb{R}^d).$$

Due to the ground-breaking work of Duchon [1], any solution of (2) has necessarily the form

$$s(x) = \sum_{j=1}^n c_j \phi_{d,k}(\|x - x_j\|) + p(x) \qquad \text{with } p \in \mathcal{P}_k^d, \tag{3}$$

where

$$\phi_{d,k}(r) = \begin{cases} r^{2k-d} \log(r) & \text{for } d \text{ even,} \\ r^{2k-d} & \text{for } d \text{ odd,} \end{cases}$$

is a fixed radial kernel function, termed *polyharmonic spline*, w.r.t. the Euclidean norm $\|\cdot\|$, and where $\mathcal{P}_k^d$ is the linear space of all d-variate polynomials of order at most k. Moreover, the coefficients $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^n$ in the major part of s in (3) are satisfying the vanishing moment conditions

$$\sum_{j=1}^n c_j p(x_j) = 0 \qquad \text{for all } p \in \mathcal{P}_k^d. \tag{4}$$

We remark that the solution s in (3) satisfying (1),(4) is unique [1,6], provided that X is $\mathcal{P}_k^d$-regular, i.e., for $p \in \mathcal{P}_k^d$ we have

$$p(x_j) = 0 \quad \text{for all } 1 \leq j \leq n \qquad \Longrightarrow \qquad p \equiv 0.$$

We shall from now assume uniqueness for s.

Our previous paper [4] analyzes the conditioning and the numerical stability of Lagrange interpolation by polyharmonic splines, where the key result in [4] is the scale-invariance of the interpolation scheme's Lagrange basis.

This paper provides a generalization of corresponding results in [4] from Lagrange interpolation to Hermite-Birkhoff reconstruction. Moreover, a suitable preconditioner is developed. To this end, the scale-invariance of the problem's condition number is proven for the more general case of Hermite-Birkhoff reconstruction.

The outline of this paper is as follows. In Section 2, we first explain Hermite-Birkhoff reconstruction by polyharmonic splines, before we show the scale-invariance of the method's Lagrange basis in Section 3. Relevant results concerning the conditioning of the reconstruction problem are proven in Section 4, before a suitable preconditioner is constructed in Section 5.

We remark that relevant applications for Hermite-Birkhoff reconstruction include particle and collocation methods for the numerical solution of PDEs as well as medical image reconstruction from X-ray scans in computerized tomography, to mention but a few. Supporting numerical examples arising from particular application scenarios are presented during the conference, where special emphasis is placed on both the variation of the sampling and the comparison with other radial kernel functions.

This work is supported by the priority program SPP 1324 of the Deutsche Forschungsgemeinschaft.

2. HERMITE-BIRKHOFF RECONSTRUCTION

To simplify notation, we fix the *order* k of $\phi_{d,k}$ and the dimension d, with assuming $2k > d$, and we let $\phi \equiv \phi_{d,k}$ and $\mathbf{BL} \equiv \mathbf{BL}^k(\mathbb{R}^d)$. To explain Hermite-Birkhoff reconstruction by polyharmonic splines, let $T = \{\tau_1, \ldots, \tau_n\}$ denote a set of linearly independent compactly supported functionals from the topological dual $\mathbf{BL}^*$ of $\mathbf{BL}$. Recall that the norm of any $\tau \in \mathbf{BL}^*$ is being defined as

$$\|\tau\| = \inf\left\{C : |\tau(f)| \leq C \cdot |f|_{\mathbf{BL}} \text{ for all } f \in \mathbf{BL}\right\}.$$

Now let us assume we are given a data vector

$$f\big|_T = (\tau_1(f), \ldots, \tau_n(f))^T \in \mathbb{R}^n$$

of irregular samples taken from $f \in \mathbf{BL}$. Reconstruction from Hermite-Birkhoff data $f\big|_T$ requires finding a reconstruction s satisfying $s\big|_T = f\big|_T$, i.e.,

$$\tau_j(s) = \tau_j(f), \qquad \text{for all } 1 \leq j \leq n. \tag{5}$$

Note that the above reconstruction problem (5) covers the interpolation problem (1). Indeed, in the special case of problem (1), the dual functionals are given by the set $T = \{\delta_{x_1}, \ldots, \delta_{x_n}\}$ of n shifted Dirac δ-distributions, where the shifts are taken about n pairwise distinct translation points in $X = \{x_1, \ldots, x_n\}$.

Now in order to generalize polyharmonic spline interpolation to polyharmonic reconstruction from Hermite-Birkhoff data, we work with a reconstruction of the form

$$s(x) = \sum_{j=1}^{n} c_j \tau_j^y(\phi(\|x-y\|)) + p(x) \qquad \text{with } p \in \mathcal{P}_k^d, \tag{6}$$

where the notation τ_j^y denotes action of dual functional $\tau_j \in T$ on variable $y \in \mathbb{R}^d$, for $1 \leq j \leq n$. Note that s in (3) coincides with s in (6), if we let $\tau_j = \delta_{x_j}$ for $1 \leq j \leq n$.

We can rewrite s in (6) by

$$s = \tau * \phi + p \qquad \text{for } \tau = \sum_{j=1}^{n} c_j \tau_j \in \mathcal{T}_k^{\perp} \text{ and } p \in \mathcal{P}_k^d, \tag{7}$$

where

$$(\tau * \phi)(x) = \tau^y(\phi(\|x-y\|))$$

denotes the convolution product between τ and $\phi(\|\cdot\|)$. Moreover, to accommodate the side condition (4) for the coefficients of the reconstruction's major part in (7), we require $\tau \in \mathcal{T}_k^{\perp}$ in (7), where

$$\mathcal{T}_k^{\perp} = \left\{\tau \in \mathbf{BL}^* : \tau(p) = 0 \text{ for all } p \in \mathcal{P}_k^d\right\}$$

is the linear space of all dual functionals from $\mathbf{BL}^*$ whose kernel contains $\mathcal{P}_k^d$.

The well-posedness of the Hermite-Birkhoff reconstruction problem (5) by using polyharmonic splines is covered as a special case in [3, Theorem 6.2], which we here quote for the reader's convenience. For more details, we refer to [3, 8].

Theorem 1. *Let $T = \{\tau_1, \ldots, \tau_n\}$ denote a set of linearly independent compactly supported linear functionals from the topological dual of $\mathbf{BL}$. Then, the reconstruction problem (5) has a solution of the form (7), where s is unique, if the set T is $\mathcal{P}_k^d$-regular, i.e., for $p \in \mathcal{P}_k^d$ we have*

$$\tau_j(p) = 0 \quad \text{for } 1 \leq j \leq n \qquad \Longrightarrow \qquad p \equiv 0.$$

■

We assume from now that T is $\mathcal{P}_k^d$-regular. In this case, the above theorem states that on given Hermite-Birkhoff data $f\big|_T$ there is one and only one element s in the *reconstruction space*

$$\mathcal{R} = \left\{ s = \tau * \phi + p : \tau = \sum_{j=1}^{n} c_j \tau_j \in \mathcal{T}_k^{\perp}, p \in \mathcal{P}_k^d \right\} \subset \mathbf{BL} \tag{8}$$

satisfying $s\big|_T = f\big|_T$. Indeed, the reconstruction $s \in \mathcal{R}$ is given by the unique orthogonal projection of $f \in \mathbf{BL}$ onto $\mathcal{R}$. Moreover, the coefficients $c \in \mathbb{R}^n$ of the major part of s and $d \in \mathbb{R}^m$ of its polynomial part p are given by the solution of the linear system

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} f\big|_T \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} \Phi &= (\tau_i^x \tau_j^y \phi(\|x-y\|))_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n} \\ P &= (\tau_j^x(x^\alpha))_{1 \leq j \leq n; |\alpha| < k} \in \mathbb{R}^{n \times m}, \end{aligned}$$

with $m = \binom{k-1+d}{d}$ denoting the dimension of $\mathcal{P}_k^d$.

Note that the well-posedness of the reconstruction problem implies that there are unique Lagrange basis functions $\lambda_1, \ldots, \lambda_n \in \mathcal{R}$ satisfying $\tau_j(\lambda_\ell) = \delta_{j\ell}$, with $\delta_{j\ell}$ denoting the usual Kronecker symbol, for $1 \leq j, \ell \leq n$. Moreover, note that for fixed $x \in \mathbb{R}^d$ the Lagrange basis functions $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x))^T \in \mathbb{R}^n$ are satisfying

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda(x) \\ \mu(x) \end{bmatrix} = \begin{bmatrix} \varphi(x) \\ \pi(x) \end{bmatrix},$$

for Lagrange multipliers $\mu(x) = (\mu_1(x), \ldots, \mu_m(x))^T \in \mathbb{R}^m$, where

$$\begin{aligned} \varphi(x) &= (\tau_j^y \phi(\|x-y\|))_{1 \leq j \leq n} \in \mathbb{R}^n \\ \pi(x) &= (x^\alpha)_{|\alpha| < k} \in \mathbb{R}^m. \end{aligned}$$

Finally, the reconstruction s in (6) can be rewritten in its Lagrange representation as

$$s(x) = \sum_{j=1}^{n} \tau_j(f) \lambda_j(x), \tag{9}$$

or,

$$s(x) = < \lambda(x) \,\big|\, f\big|_T >,$$

in short hand notation, where $< \cdot \big| \cdot >$ denotes the inner product of the Euclidean space $\mathbb{R}^n$.

3. SCALE-INVARIANCE OF THE POLYHARMONIC SPLINE RECONSTRUCTION SCHEME

In this section, we show the scale-invariance of the polyharmonic spline reconstruction scheme. To this end, let us regard, for fixed $h > 0$, the *scaled reconstruction problem*

$$\tau_j(\sigma_h(s^h)) = \tau_j(\sigma_h(f)) \qquad \text{for all } 1 \le j \le n, \tag{10}$$

where σ_h is the dilatation operator, defined as $\sigma_h(f) = f(\cdot/h)$.

By Theorem 1 there is a unique polyharmonic spline reconstruction of the form $s \equiv \sigma_h(s^h) \in \mathcal{R}$ satisfying (10), or, if we solve (10) for s^h, we obtain $s^h = \sigma_h^{-1}(s)$. Let us collect all possible solutions s^h to (10) in the scaled reconstruction space

$$\mathcal{R}^h = \left\{ s^h = \sigma_h^{-1}(s) : s \in \mathcal{R} \right\}.$$

The following theorem, which can be viewed as a generalization of our previous result [4, Lemma 3.2], states that $\mathcal{R}^h$ is a scaled version of $\mathcal{R}$, or, in other words, the polyharmonic spline reconstruction scheme is scale-invariant.

Theorem 2. *The reconstruction space $\mathcal{R}$ in (8) is scale-invariant, i.e., $\mathcal{R}^h = \mathcal{R}$ for any $h > 0$.*

Proof: If the space dimension d is odd, then the identity $\mathcal{R} = \sigma_h(\mathcal{R})$ immediately follows from the homogeneity of the polyharmonic spline $\phi \equiv \phi_{d,k}$ and the linearity of the dual functionals in T.

Now suppose d is even. In this case we have

$$\phi_{d,k}(hr) = h^{2k-d} \left(\phi_{d,k}(r) + r^{2k-d} \log(h) \right).$$

Therefore, any $s^h \in \mathcal{R}^h$ has, up to some $p \in \mathcal{P}_k^d$, the form

$$s^h(hx) = h^{2k-d} \left(\sum_{j=1}^{n} c_j \tau_j^y (\phi_{d,k}(\|x-y\|)) + \log(h) q(x) \right),$$

where we let

$$q(x) = \sum_{j=1}^{n} c_j \tau_j^y \left(\|x-y\|^{2k-d} \right).$$

To see that s^h is contained in $\mathcal{R}$, it remains to show that the degree of the polynomial q is at most $k-1$. To this end, we rewrite q as

$$\begin{aligned} q(x) &= \sum_{j=1}^{n} c_j \tau_j^y \left(\sum_{|\alpha|+|\beta|=2k-d} c_{\alpha,\beta} x^\alpha y^\beta \right) \\ &= \sum_{|\alpha|+|\beta|=2k-d} c_{\alpha,\beta} x^\alpha \sum_{j=1}^{n} c_j \tau_j^y (y^\beta) \end{aligned}$$

for some coefficients $c_{\alpha,\beta}$ with $|\alpha| + |\beta| = 2k - d$. Now due to the orthogonality relation $\sum_{j=1}^n c_j \tau_j \in \mathcal{T}_k^\perp$, as required in (7), this implies that the degree of the polynomial q is at most $2k - d - k = k - d < k$ for $d \ge 1$. Therefore, $s^h \in \mathcal{R}$, and so $\mathcal{R}^h \subset \mathcal{R}$. The converse inclusion $\mathcal{R} \subset \mathcal{R}^h$ can be proven accordingly. Altogether, we find $\mathcal{R} = \sigma_h(\mathcal{R}^h)$ for any $h > 0$ in any space dimension d, which completes our proof. ∎

Due to the well-posedness of the (scaled or unscaled) reconstruction problem, in $\mathcal{R}^h$ or $\mathcal{R}$, the (unique) Lagrange basis functions are also scale-invariant. Hence, the following result can be viewed as a direct consequence of the previous theorem.

Corollary 1. *The Lagrange basis functions $\lambda = (\lambda_1^h, \ldots, \lambda_n^h)^T$ of $\mathcal{R}^h$, satisfying*

$$\tau_j(\sigma_h(\lambda_\ell^h)) = \delta_{j\ell} \qquad \text{for } 1 \le j, \ell \le n,$$

are scale-invariant. More precisely, for any $h > 0$ we have

$$\lambda = \sigma_h(\lambda^h)$$

for the Lagrange basis functions $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ of $\mathcal{R}$. ∎

4. CONDITIONING OF THE RECONSTRUCTION

This and the following section are concerning the stability of the polyharmonic spline reconstruction scheme, where our aim is to construct a numerically stable algorithm for the evaluation of any polyharmonic spline reconstruction $s \in \mathcal{R}$. To this end, we first analyze the conditioning of the given problem (done in this section), before we turn to the construction of a stable evaluator in the following section. For a comprehensive discussion on the relevant principles and concepts from error analysis, especially the *condition number* of a given *problem* versus the *stability* of a *numerical algorithm*, we recommend the textbook [2].

To discuss the conditioning of polyharmonic spline reconstruction, recall that the condition number of the reconstruction problem is given by the operator norm of the reconstruction operator $\mathcal{I} : \mathbf{BL} \to \mathcal{R}$, which returns on any argument $f \in \mathbf{BL}$ a (unique) polyharmonic spline reconstruction $s \in \mathcal{R}$ satisfying $f\big|_T = s\big|_T$.

Therefore, for fixed T, the condition number of polyharmonic spline reconstruction from data $f\big|_T$ is the smallest number $\kappa \equiv \kappa(T)$ satisfying

$$|\mathcal{I}f|_{\mathbf{BL}} \le \kappa \cdot |f|_{\mathbf{BL}} \qquad \text{for all } f \in \mathbf{BL}.$$

The following result is useful for the subsequent discussion on the stability of polyharmonic spline reconstruction.

Theorem 3. *The condition number κ of polyharmonic spline reconstruction is bounded above by the Lebesgue constant*

$$\Lambda \equiv \Lambda(T) = \sum_{j=1}^{n} \|\tau_j\| \cdot |\lambda_j|_{\mathbf{BL}},$$

i.e., $\kappa \le \Lambda$.

Proof: Let for any $f \in \mathbf{BL}$ and for fixed $T = \{\tau_1, \ldots, \tau_n\}$ the data vector $f\big|_T$ be given, and let $s = \mathcal{I}(f) \in \mathcal{R}$ denote the polyharmonic spline reconstruction of the form (7) satisfying

$f\big|_T = s\big|_T$. Using the Lagrange representation of s in (9), we immediately obtain the estimate

$$\begin{aligned} |\mathcal{I}(f)|_{\mathbf{BL}} &= |s|_{\mathbf{BL}} \leq \sum_{j=1}^{n} |\tau_j(f)| \cdot |\lambda_j|_{\mathbf{BL}} \\ &\leq \left(\sum_{j=1}^{n} \|\tau_j\| \cdot |\lambda_j|_{\mathbf{BL}} \right) |f|_{\mathbf{BL}}, \end{aligned}$$

and therefore $\kappa \leq \Lambda$. ■

Note that by $1 = |\tau_j(\lambda_j)| \leq \|\tau_j\| \cdot |\lambda_j|_{\mathbf{BL}}$, for $1 \leq j \leq n$, we find $\Lambda \geq n$, and so the Lebesgue constant is bounded below by the number n of functionals in T. The following result is a direct consequence of the scale-invariance of the Lagrange basis.

Corollary 2. *The Lebesgue constant of polyharmonic spline reconstruction is scale-invariant.*

Proof: For fixed $T = \{\tau_1, \ldots, \tau_n\}$ and any $h > 0$, let

$$T \circ \sigma_h = \{\tau_1 \circ \sigma_h, \ldots, \tau_n \circ \sigma_h\}.$$

Then, the identity

$$\Lambda(T \circ \sigma_h) = \sum_{j=1}^{n} \|\tau_j\| \cdot |\sigma_h(\lambda_j^h)|_{\mathbf{BL}} = \sum_{j=1}^{n} \|\tau_j\| \cdot |\lambda_j|_{\mathbf{BL}} = \Lambda(T)$$

holds, where we used the scale-invariance of the Lagrange basis functions, $\lambda = \sigma_h(\lambda^h)$, from Corollary 1. ■

5. PRECONDITIONING

A naive method for solving the scaled reconstruction problem is given by the direct solution of the scaled linear system

$$\left[\begin{array}{cc} \Phi_h & P_h \\ P_h^T & 0 \end{array} \right] \left[\begin{array}{c} c^h \\ d^h \end{array} \right] = \left[\begin{array}{c} \sigma_h(f)\big|_T \\ 0 \end{array} \right], \qquad (11)$$

where

$$\begin{aligned} \Phi_h &= (\tau_i^x \tau_j^y \phi(\|(x-y)/h\|))_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n} \\ P_h &= (\tau_j^x((x/h)^\alpha))_{1 \leq j \leq n; |\alpha| < k} \in \mathbb{R}^{n \times m}. \end{aligned}$$

It is convenient to abbreviate the system (11) as

$$A_h b^h = f^h. \qquad (12)$$

Now let us turn to the construction of a numerically stable algorithm for evaluating the polyharmonic spline reconstruction s^h satisfying (10). To this end, we require that the given reconstruction problem (10) is well-conditioned. Note that according to Corollary 2, the Lebesgue constant Λ, being an upper bound of the reconstruction problem's condition number κ, merely depends on the functionals in T, but not on the scale h.

However, the spectral condition number $\kappa_2(A_h)$ of matrix A_h in (12) depends on h. The following rescaling can be viewed as a simple way of preconditioning the matrix A_h for very small h, where $\kappa_2(A_h) \gg \kappa_2(A_1)$. Our method relies on the following sequence of calculations, where we use the scale-invariance of the Lagrange basis functions, $\lambda^h = \sigma_h(\lambda)$, from Corollary 1,

$$\begin{aligned} s^h(hx) &= < \lambda^h(hx) \,\big|\, f^h > = < \lambda(x) \,\big|\, f^h > \\ &= < (\lambda(x), \mu(x))^T \,\big|\, (f^h, 0)^T > \\ &= < A_1^{-1}(\varphi(x), \pi(x))^T \,\big|\, (f^h, 0)^T > \\ &= < (\varphi(x), \pi(x))^T \,\big|\, A_1^{-1}(f^h, 0)^T >, \end{aligned}$$

which yields the more suitable representation

$$s^h(hx) = < \beta_1(x) \,\big|\, A_1^{-1} \cdot f_h > \qquad (13)$$

for the polyharmonic spline reconstruction, where we let $\beta_1(x) = (\varphi(x), \pi(x))^T$ and $f_h = (f^h, 0)^T$. Due to representation (13) we can evaluate s^h at hx by solving the system $A_1 \cdot \mathbf{b} = f_h$, whose solution $\mathbf{b} \in \mathbb{R}^{n+q}$ yields the coefficients of $s^h(hx)$ w.r.t. the basis functions in $\beta_1(x)$.

Finally, note that by working with the representation (13) for s^h, we can avoid solving the linear system (11). This is useful insofar as the linear system (11) is often ill-conditioned for very small h, but well-conditioned for sufficiently large h. Hence, for the sake of numerical stability, one should, for small h, avoid solving (11) directly (cf. [7] for details). Further supporting arguments on this, along with illustrative numerical examples and comparisons, are presented during the conference.

6. REFERENCES

[1] J. Duchon: Splines minimizing rotation-invariant semi-norms in Sobolev spaces. In: *Constructive Theory of Functions of Several Variables*, W. Schempp and K. Zeller (eds.), Springer, Berlin, 1977, 85–100.

[2] N.J. Higham: *Accuracy and Stability of Numerical Algorithms.* SIAM, Philadelphia, 1996.

[3] A. Iske: Reconstruction of functions from generalized Hermite-Birkhoff data. In: *Approximation Theory VIII.* World Scientific, Singapore, 1995, 257–264.

[4] A. Iske: On the approximation order and numerical stability of local Lagrange interpolation by polyharmonic splines. In: *Modern Developments in Multivariate Approximation.* Birkhäuser, Basel, 2003, 153–165.

[5] A. Iske: *Multiresolution Methods in Scattered Data Modelling.* Springer-Verlag, Berlin, 2004.

[6] C.A. Micchelli: Interpolation of scattered data: distance matrices and conditionally positive definite functions. Constr. Approx. **2** (1986), 11–22.

[7] F.J. Narcowich and J.D. Ward: Norm estimates for the inverses of a general class of scattered-data radial-function interpolation matrices. J. Approx. Th. **69**, 1992, 84–109.

[8] Z. Wu: Hermite-Birkhoff interpolation of scattered data by radial basis functions. Appr. Th. Appl. **8**(2), 1992, 1–10.