RECONSTRUCTION OF SMOOTH SIGNALS FROM IRREGULAR SAMPLES BY USING RADIAL BASIS FUNCTION APPROXIMATION

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ABSTRACT

Based on constrained least squares approximation and techniques from computational geometry, this paper shall provide a new approximation scheme which achieves to combine well-known features from radial basis function interpolation with recent developments concerning scattered data filtering.

The suggested method is shown to be well-posed, and moreover by using local error estimates from radial basis function interpolation, approximation orders for the least squares error are proven. An efficient adaptive algorithm for the purpose of automatically selecting a suitable basis of the approximation space is provided.

1. INTRODUCTION

Radial basis functions are well-established tools for multivariate scattered data interpolation. For a brief discussion of the basic features of radial basis function interpolation, let

$$T_Z(f) = (f(z_1), \ldots, f(z_M))^T \in \mathbf{R}^M$$

denote a given data vector containing irregular samples from an unknown signal $f : \mathbf{R}^d \to \mathbf{R}$, $d > 1$, collected at a set $Z = \{z_1, \ldots, z_M\} \subset \mathbf{R}^d$ containing finitely many pairwise distinct scattered locations. Due to the radial basis function interpolation scheme, for a fixed continuous *basis function* $\phi : [0, \infty) \to \mathbf{R}$, the solution $s : \mathbf{R}^d \to \mathbf{R}$ of the interpolation problem $T_Z(f) = T_Z(s)$ is required to possess the form

$$s = \lambda_Z^y \phi(\| \cdot - y\|) + p \quad \lambda_Z \in \mathcal{L}_m^\perp, p \in \mathcal{P}_m^d. \qquad (1)$$

Here, $\| \cdot \|$ is the Euclidean norm on $\mathbf{R}^d$, $\mathcal{P}_m^d$ stands for the linear space containing all d-variate real-valued polynomials of order m or less and the functional

$$\lambda_Z = \sum_{z \in Z} \lambda_z \delta_z,$$

whose action on the variable y in (1) is denoted by λ_Z^y, is required to be an element from the linear space

$$\mathcal{L}_m^\perp = \left\{ \lambda = \sum_{x \in X} \lambda_x \delta_x \; : \; X \subset \mathbf{R}^d, |X| < \infty, \lambda\big|_{\mathcal{P}_m^d} \equiv 0 \right\}$$

containing all finite linear combinations of the Dirac δ-functional which are vanishing on $\mathcal{P}_m^d$.

Under these presuppositions, the above interpolation problem $T_Z(f) = T_Z(s)$ is guaranteed to possess a solution provided that the basis function ϕ is conditionally positive definite of order m on $\mathbf{R}^d$.

Definition 1.1: *A function $\phi : [0, \infty) \to \mathbf{R}$ is said to be* `conditionally positive definite` *on $\mathbf{R}^d$, $\phi \in \mathtt{cpd}(m, d)$, iff for all possible choices of functionals $\lambda \in \mathcal{L}_m^\perp \setminus \{0\}$ the quadratic form*

$$\lambda^x \lambda^y \phi(\|x - y\|)$$

is positive.

The interpolant s will be unique, if the points from Z are $\mathcal{P}_m^d$*-nondegenerate*, i.e.

$$p(z) = 0 \text{ for all } z \in Z \quad \Longrightarrow \quad p \equiv 0$$

holds true for all $p \in \mathcal{P}_m^d$.

Indeed, due to [8], this then guarantees the collocation matrix of the interpolation problem to be nonsingular.

Theorem 1.2: *Let $X = \{x_1, \ldots, x_N\} \subset \mathbf{R}^d$ be a $\mathcal{P}_m^d$-nondegenerate set containing finitely many pairwise distinct points, and let $p_1, \ldots, p_Q$ be a basis of $\mathcal{P}_m^d$, i.e. $Q = \binom{m-1+d}{d}$. Then, the matrix*

$$A = \begin{bmatrix} P_X^T & 0 \\ A_X & P_X \end{bmatrix} \in \mathbf{R}^{(N+Q)\times(N+Q)},$$

where

$$A_X = (\phi(\|x_j - x_k\|))_{1 \le j,k \le N} \text{ and } P_X = (p_\ell(x_k))_{\substack{1 \le k \le N \\ 1 \le \ell \le Q}}$$

is nonsingular provided that $\phi \in \mathtt{cpd}(m, d)$. ■

A list of radial functions which are conditionally positive definite on $\mathbf{R}^d$ for all space dimensions, can be found in the table below.

Thin Plate Splines	$\phi_{\rm TPS}(r)$	$= r^2 \log(r)$
Multiquadrics	$\phi_{\rm MQ}(r)$	$= -(r^2+1)^{1/2}$
Gaussians	$\phi_{\rm G}(r)$	$= e^{-r^2}$
Inverse Multiquadrics	$\phi_{\rm IMQ}(r)$	$= (r^2+1)^{-1/2}$

As to their corresponding order m of conditionally positive definiteness, we find that $\phi_{\rm TPS} \in \mathtt{cpd}(2,d)$, $\phi_{\rm MQ} \in \mathtt{cpd}(1,d)$, and $\phi_{\rm G}, \phi_{\rm IMQ} \in \mathtt{cpd}(0,d)$. Recent developments [13] have provided compactly supported positive definite radial functions, one is $\phi_{\rm W}(r) = (1-r/2)_+^4(2r+1) \in \mathtt{cpd}(0,3)$. For a comprehensive treatment of radial basis functions, in particular for a complete list of all common functions from $\mathtt{cpd}(m,d)$, we refer the reader to the survey paper [9].

A major drawback, however, of radial basis function interpolation are the prohibitive computational costs for large data sets. Moreover, the condition number of the collocation matrix blows up whenever the *separation distance*

$$q_Z = \min_{1 \le j < k \le M} \|z_j - z_k\|$$

of the set Z is small. Finally, each evaluation of the interpolant takes $\mathcal{O}(M)$ time for basis functions which are not compactly supported, and this is for large numbers M an awkward restriction for applications which require frequent evaluations of the solution.

This paper suggests a new approximation scheme which provides for a $\mathcal{P}_m^d$-nondegenerate subset $X = \{x_1, \ldots, x_N\}$ of the set Z with $Q < N \ll M$ an approximation s^* to the unknown signal f by minimizing $\|T_Z(s-f)\|$ among all functions from the *approximation space*

$$\mathcal{S}_{\phi,X} = \left\{ s = \lambda_X^y \phi(\|\cdot - y\|) + p \,:\, \lambda_X \in \mathcal{L}_m^\perp, p \in \mathcal{P}_m^d \right\}.$$

The well-posedness of the approximation scheme is proven and error bounds for the *least squares error* $E_{X,Z}(f) = \|T_Z(s^* - f)\|$ are provided by using well-known local error estimates from radial basis function interpolation. A technique for automatically selecting a suitable subset $X \subset Z$ from the given set Z is presented.

2. LEAST SQUARES APPROXIMATION

In this section, we focus on solving the linear least squares problem

$$\min_{s \in \mathcal{S}_{\phi,X}} \|T_Z(s-f)\|. \tag{2}$$

Strictly speaking, by the linear constraints on λ_X in the definition of $\mathcal{S}_{\phi,X}$ the above problem (2) is a *linear least squares problem with linear equality constraints*.

We will show that under the assumptions from the previous section, the problem (2) possesses a unique solution s^* which can directly be computed by using standard techniques. But this requires some preparation. Without loss of generality we assume that the points in Z are ordered such that the points from $X \subset Z$ come first, i.e. $x_k = z_k$, $1 \le k \le N$, and we let $Y = Z \setminus X$. Moreover, we shall use the abbreviation

$$A_{X,Z} = (\phi(\|z_k - x_j\|))_{\substack{1 \le k \le M \\ 1 \le j \le N}} \in \mathbf{R}^{M \times N}.$$

In particular, $A_{X,X} = A_X$.

Theorem 2.1: *Let X be a $\mathcal{P}_m^d$-nondegenerate subset of Z and let $\phi \in \mathtt{cpd}(m,d)$. Then, the above constrained linear least squares problem (2) has a unique solution.*

Proof: The above linear least squares problem (2) can be rewritten as

$$\|Bw - T_Z(f)\| \to \min_{u^T P_X = 0} ! \tag{3}$$

where the vector $w = [u^T | v^T]^T \in \mathbf{R}^{N+Q}$ of unknowns is split into vectors $u \in \mathbf{R}^N$ for the coefficients of the major part $\lambda_X^y \phi(\|\cdot - y\|)$ and $v \in \mathbf{R}^Q$ for the polynomial part. Consequently, the matrix

$$B = \left[\begin{array}{c|c} A_{X,Z} & P_Z \end{array}\right] \in \mathbf{R}^{M \times (N+Q)}$$

contains the evaluations of the basis functions at the points from Z. Note that the constraints $u^T P_X = 0$ in (3) can by expansion be rewritten as $Cw = 0$ where

$$C = \left[\begin{array}{c|c} P_X^T & 0 \end{array}\right] \in \mathbf{R}^{Q \times (N+Q)}.$$

According to [6], Chapter 20, Theorem 20.9, the problem (3) has a solution provided that the matrix C has full rank. Moreover, under the assumption $\mathrm{rank}(C) = Q$ a solution of (3) will be unique if and only if the augmented matrix

$$D = \left[\begin{array}{c} C \\ B \end{array}\right] \in \mathbf{R}^{(M+Q) \times (N+Q)}$$

has full rank, i.e. $\mathrm{rank}(D) = N + Q$.

Indeed, by splitting the matrix D as

$$D = \left[\begin{array}{cc} P_X^T & 0 \\ A_X & P_X \\ \hline A_{X,Y} & P_Y \end{array}\right]$$

and by using Theorem 1.2, we see that $\mathrm{rank}(D) = N + Q$ holds true. Finally, C has full rank due to the injectivity of P_X which is a direct consequence from our assumption on X. ∎

Solving (3) by using direct elimination requires according to [6] a partitioning of

$$D = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline B_1 & B_2 \end{array}\right] \tag{4}$$

with $C_1 \in \mathbf{R}^{Q\times Q}, C_2 \in \mathbf{R}^{Q\times N}, B_1 \in \mathbf{R}^{M\times Q}, B_2 \in \mathbf{R}^{M\times N}$, and

$$w = \left[\begin{array}{c} w_1 \\ w_2 \end{array}\right]$$

with $w_1 \in \mathbf{R}^Q, w_2 \in \mathbf{R}^N$. Without loss of generality we assume that C_1 is nonsingular which is equivalent to require that the set $X_Q = \{x_1, \dots, x_Q\} \subset X$ containing the first Q points of X is $\mathcal{P}_m^d$-nondegenerate. Since X itself was supposed to be $\mathcal{P}_m^d$-nondegenerate, it follows that X contains a $\mathcal{P}_m^d$-nondegenerate subset of size Q.

Due to the given constraints $Cw = 0$, we find that $w_1 = -C_1^{-1}C_2w_2$ holds, and therefore by using (4) we have

$$\|Bw - T_Z(f)\| = \|\tilde{B}_2 w_2 - T_Z(f)\|$$

with $\tilde{B}_2 = B_2 - B_1C_1^{-1}C_2$. Consequently, the computation of the solution of (3) can be reduced to solving the unconstrained linear least squares problem

$$\|\tilde{B}_2 w_2 - T_Z(f)\| \to \min_{w_2 \in \mathbf{R}^N} ! \tag{5}$$

A practical way for computing the solution of (5) and thus of (3) is dating back to [1] (see also [6], Chapter 21). The starting point in [1] is a decomposition of $C = Q_1^T[\tilde{C}_1, \tilde{C}_2]$ where $Q_1 \in \mathbf{R}^{Q\times Q}$ is orthogonal and $\tilde{C}_1 \in \mathbf{R}^{Q\times Q}$ is upper triangular. Using the identity

$$\tilde{B}_2 = B_2 - (B_1\tilde{C}_1^{-1})(Q_1C_2) = B_2 - \tilde{B}_1\tilde{C}_2$$

the computation of $\tilde{B}_2$ requires to solve the triangular system $\tilde{B}_1\tilde{C}_1 = B_1$ for the determination of $\tilde{B}_1 \in \mathbf{R}^{M\times Q}$. For solving (5) it remains to triangularize the matrix $\tilde{B}_2 \in \mathbf{R}^{M\times N}$ by finding an orthogonal matrix $Q_2 \in \mathbf{R}^{M\times M}$ such that

$$Q_2 \cdot [\tilde{B}_2 | T_Z(f)] = \left[\begin{array}{c|c} \hat{B}_2 & g_1 \\ \hline 0 & g_2 \end{array}\right] \in \mathbf{R}^{M\times(N+1)}$$

where $\hat{B}_2 \in \mathbf{R}^{N\times N}$ is upper triangular, and $g_1 \in \mathbf{R}^N, g_2 \in \mathbf{R}^{M-N}$. The computation of the solution w of (3) can then be accomplished by solving the two triangular systems

$$\begin{aligned} \hat{B}_2 w_2 &= g_1 \\ \tilde{C}_1 w_1 &= -\tilde{C}_2 w_2 \end{aligned}$$

one after the other.

3. APPROXIMATION ORDERS

The purpose of this section is to provide convergence rates for the least squares error $E_{X,Z}(f)$ by using well-known local error estimates from radial basis function interpolation. For the convenience of the reader we shall briefly compose the relevant material. More background can be found in Schaback's survey [9] and his recent papers [10, 11, 12].

For a fixed $\phi \in \texttt{cpd}(m,d)$, let s_I denote the unique function from $\mathcal{S}_{\phi,X}$ satisfying $T_X(f) = T_X(s_I)$. Then, for all functions $f : \mathbf{R}^d \to \mathbf{R}$ from the basis function's *native space* $\mathcal{F}_\phi$, the error of the interpolation can be bounded by an estimate of the form

$$|s_I(x) - f(x)| \le |f|_\phi \cdot P_{X,\phi}(x)$$

where the *power function* $P_{X,\phi}(x)$ is pointwise given by the norm of the error functional at $x \in \mathbf{R}^d$. The space $\mathcal{F}_\phi$ contains $\mathcal{P}_m^d$ and all functions $f : \mathbf{R}^d \to \mathbf{R}$ which can be recovered via

$$f(x) = (2\pi)^{-d} \int_{\mathbf{R}^d} e^{ix^T\omega} \hat{f}(\omega)\, d\omega$$

and which are *smooth* by $\hat{f}/\sqrt{\hat{\phi}} \in L_2(\mathbf{R}^d)$. Here, $\hat{\phi}$ denotes the basis function's (generalized) d-variate Fourier transform which is for all common $\phi \in \texttt{cpd}(m,d)$ positive almost everywhere. By

$$|f|_\phi^2 = (2\pi)^{-d} \int_{\mathbf{R}^d} \frac{|\hat{f}(\omega)|^2}{\hat{\phi}(\omega)}\, d\omega$$

the space $\mathcal{F}_\phi$ is equipped with a (semi-)norm $|\cdot|_\phi$ with nullspace $\mathcal{P}_m^d$, and $\mathcal{F}_\phi/\mathcal{P}_m^d$ is a Hilbert space. Further details on the structure of the spaces $\mathcal{F}_\phi$ can be found in [11].

According to [7, 15], the power function itself can for all of the radial basis function from the introduction be bounded above by

$$P_{X,\phi}(x) \le F_\phi(h_{\rho,X}(x))$$

where for a positive radius $\rho > 0$, the quantity

$$h_{\rho,X}(x) = \max_{y\in B_\rho(x)} \min_{1\le j\le N} \|y - x_j\|$$

reflects the local density of the points from X around x, and $F_\phi : \mathbf{R}_{\ge 0} \to \mathbf{R}_{\ge 0}$ is a specific monotonic function with $F_\phi(0) = 0$. The functions F_ϕ and the (generalized) Fourier transforms $\hat{\phi}$ can (up to a positive factor) be found in the table below, where K_ν denotes the modified Bessel function. A complete list of all common radial basis functions along with their Fourier transforms and corresponding functions F_ϕ can be found in [12], and for compactly supported radial basis functions see [14].

ϕ	$\hat{\phi}(s) \doteq$	$F_\phi^2(h) \doteq$
ϕ_{TPS}	s^{-d-2}	h^2
ϕ_{MQ}	$-K_{(d+1)/2}(s) \cdot s^{-(d+1)/2}$	$e^{-\alpha/h}, \alpha > 0$
ϕ_{G}	$e^{-s^2/4}$	$e^{-\alpha/h^2}, \alpha > 0$
ϕ_{IMQ}	$K_{(d-1)/2}(s) \cdot s^{-(d-1)/2}$	$e^{-\alpha/h}, \alpha > 0$

In summary, we obtain

Theorem 3.1: *Let $\phi \in \{\phi_{\mathrm{TPS}}, \phi_{\mathrm{MQ}}, \phi_{\mathrm{G}}, \phi_{\mathrm{IMQ}}, \phi_{\mathrm{W}}\}$, and $\rho > 0$. Then, there exists a constant $h_0 > 0$, such that for all $x \in \mathbf{R}^d$ satisfying $h_{\rho,X}(x) < h_0$, the error bound*

$$|f(x) - s_I(x)| \leq |f|_\phi \cdot F_\phi(h_{\rho,X}(x)) \tag{6}$$

holds for all $f \in \mathcal{F}_\phi$. ■

In situations where $\hat{\phi}$ satisfies for some $s_\infty > 0$ and suitable constants $0 < c_1 \leq c_2$ the decay condition

$$c_1 \|\omega\|^{-d-s_\infty} \leq \hat{\phi}(\|\omega\|) \leq c_2 \|\omega\|^{-d-s_\infty} \tag{7}$$

for $\|\omega\| \to \infty$, it turns out that $F_\phi(h) \doteq h^{s_\infty/2}$. The basis functions ϕ_{TPS} and ϕ_{W} are two possible examples. The latter satisfies (7) for $d = 3$ with $s_\infty = 3$, and consequently $F_{\phi_{\mathrm{W}}}(h) \doteq h^{3/2}$ (cf. [14]).

By simple conclusion, it is straightforward to carry over the local error estimate (6) for bounding the least squares error $E_{X,Z}(f)$ in terms of the density

$$h_{\rho,X,Y} = \max_{y \in Y} h_{\rho,X}(y)$$

of the set X around the points from Y.

Corollary 3.2: *Let ϕ and ρ be as in Theorem 3.1. Then, there exists a constant $h_0 > 0$, such that for all subsets $X \subset Z$ satisfying $h_{\rho,X,Y} < h_0$ the least squares error $E_{X,Z}(f)$ can be bounded by*

$$\|T_Z(s^* - f)\| \leq |f|_\phi \cdot \sqrt{M - N} \cdot F_\phi(h_{\rho,X,Y})$$

for all functions $f \in \mathcal{F}_\phi$.

Proof: Since $\|T_Z(s^* - f)\| \leq \|T_Z(s_I - f)\|$ and by using Theorem 3.1 we find that

$$\begin{aligned} \|T_Z(s^* - f)\|^2 &\leq \sum_{k=N+1}^{M} |s_I(z_k) - f(z_k)|^2 \\ &\leq |f|_\phi^2 \cdot \sum_{k=N+1}^{M} F_\phi^2(h_{\rho,X}(z_k)) \\ &\leq |f|_\phi^2 \cdot (M - N) \cdot F_\phi^2(h_{\rho,X,Y}) \end{aligned}$$

holds true which completes our proof. ■

4. CHOICE OF THE APPROXIMATION SPACE

The success of the approximation scheme from Section 2 depends on both the choice of the basis function ϕ and the selection of a suitable subset $X \subset Z$. The error bounds from the previous section suggest to select a subset $X \subset Z$ whose distance $h_{X,Y} = h_{0,X,Y}$ to $Y = Z \setminus X$ is as small as possible. The selection of an *optimal* subset $X \subset Z$ would require to minimize $h_{X,Y}$ among all subsets of size N. However, apart from special cases such as $N = 1, 2$, this optimization problem is NP-complete [4]. Therefore, it is appropriate to approximate an optimum by using a polynomial algorithm. For the purposes of this paper, we prefer to work with a recent approximation algorithm [5] which has been shown to be efficient and effective for automatically selecting a subset $X \subset Z$ with a small value of $h_{X,Y}$. The algorithm is a combination of *Thinning* and *Exchange*. Whilst the class of Thinning Algorithms has been introduced in [3] and shown to be useful for hierarchical radial basis function interpolation in [2], the Exchange Algorithm was especially tailored for decreasing the quantity $h_{X,Y}$ for any given $X \subset Z$ by gradually exchanging pairs of points from the sets X and $Y = Z \setminus X$.

4.1. Thinning

Given a number $N \ll M$, a subset $X \subset Z$ of size N is selected by gradually removing $M - N$ points from Z. At each step, a *removable* point is deleted from the current subset.

Definition 4.1: *A point $x \in X$ is said to be* `removable` *from X, if $q_X = \min_{\tilde{x} \in X \setminus \{x\}} \|x - \tilde{x}\|$, i.e. the minimum q_X is attained at $x \in X$.*

Thinning Algorithm:

(1) **Input:** $Z_M = Z$ with $|Z| = M$.

(2) **FOR** $k = M, \ldots, N+1$

  (a) locate a removable point $z \in Z_k$

  (b) let $Z_{k-1} = Z_k \setminus \{z\}$

(3) **Output:** $X = Z_N \subset Z$ with $|X| = N$.

By the above definition, a removable point is not unique which could be compensated by suitable side conditions, as some of them are suggested in [3]. However, in [3] we found that this does in general situations not significantly change the result of the Thinning Algorithm's output.

Note that the Thinning Algorithm is *greedy*, i.e. at each single step k, $N+1 \leq k \leq M$, it yields an optimal subset $Z_{k-1} \subset Z_k$ of Z_k.

As to the computational costs of the Thinning Algorithm, in [3] we have shown that the complexity is asymptotically $\mathcal{O}(M^3)$ for each space dimension d, whereas for the special case $d = 2$ by making use of Delaunay triangulations and heap techniques we have implemented a Thinning Algorithm at average computational costs of $\mathcal{O}(M \log(M))$.

4.2. Exchange

Given a subset $X_{\mathrm{T}} \subset Z$ output by the Thinning Algorithm, the Exchange Algorithm will in a postprocessing step serve to compute a *locally optimal* subset $X_{\mathrm{E}} \subset Z$ of size $N = |X_{\mathrm{T}}|$.

Definition 4.2: *Let $X \subset Z$ and $Y = Z \setminus X$. We say that a pair $(x, y) \in X \times Y$ is* `exchangeable`*, if*

$$h_{X,Y} > h_{(X\setminus\{x\})\cup\{y\},(Y\setminus\{y\})\cup\{x\}}$$

holds true. The subset $X \subset Z$ is said to be `locally optimal` *w.r.t. Z if there is no exchangeable pair of points in $X \times Y$.*

Exchange Algorithm:

(1) **Input:** $X \subset Z$

(2) **WHILE** (X not locally optimal w.r.t. Z)

  (a) locate exchangeable pair $(x, y) \in X \times Y$

  (b) let $X = (X \setminus \{x\}) \cup \{y\}$ and
  $Y = (Y \setminus \{y\}) \cup \{x\}$

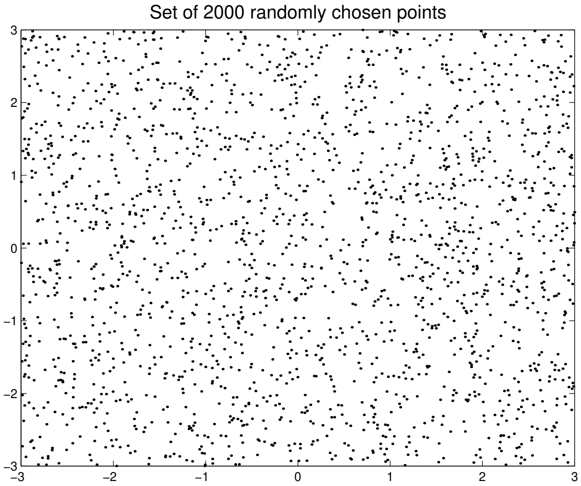
(3) **Output:** X, locally optimal w.r.t. Z.

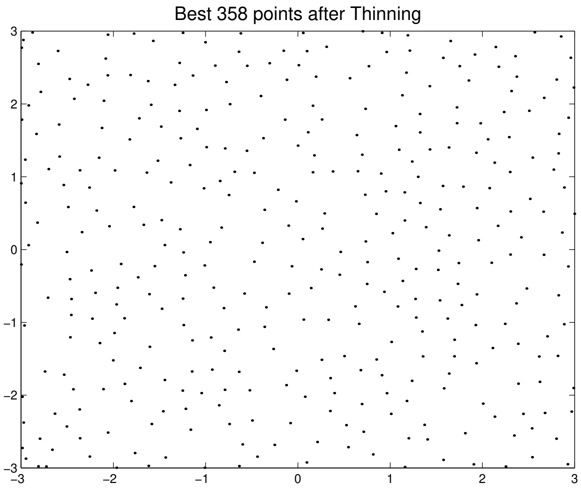
Note that the Exchange Algorithm terminates after finitely many steps since Z was assumed to be finite and after each exchange the (nonnegative) value of $h_{X,Y}$ is strictly decreasing.

Again, by employing heap techniques, Delaunay triangulations and their dual Voronoi diagrams we have implemented an Exchange Algorithm for the special case of two dimensions by using the *sufficient* criterion for an exchangeable pair from [5]. This requires $\mathcal{O}(M)$ operations for the setup, whereas each exchange costs $\mathcal{O}(1)$ for the best case and $\mathcal{O}(M)$ for the worst case. Therefore, Exchange is as cheap as Thinning in situations where the number n_{E} of exchanges is approximately $\log(M)$.

5. NUMERICAL EXAMPLE

For the purpose of illustration let us consider a small example with a scattered data set $Z \subset [-3, 3]^2$ containing $M = 2000$ randomly chosen points. We chose $N = 358$ for extracting the subset X_{T} from Z by using Thinning.





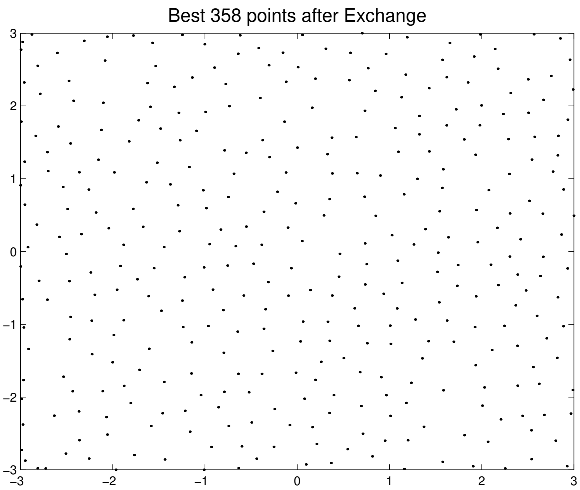
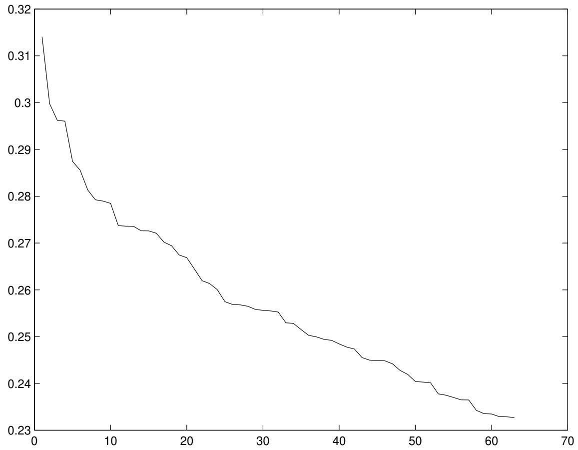


Figure 1: Graph of $h_{X,Y}$.



The subsets X_{T} generated by Thinning and X_{E} output by Exchange after $n_{\mathrm{E}} = 62$ steps are along with the set Z displayed in the figures on the previous page. At first sight, the spatial distributions of the points from $X_{\mathrm{T}}, X_{\mathrm{E}}$ look quite similar. However, we found that by the Exchange the quantity $h_{X,Y}$ drops down by about one third from $h_{X_{\mathrm{T}},Y_{\mathrm{T}}} = 0.314082$ to $h_{X_{\mathrm{E}},Y_{\mathrm{E}}} = 0.232700$, cf. Figure 1. For comparing X_{T} against X_{E}, we have generated a data vector $T_Z(f)$ by sampling MATLAB's *peaks function*

$$\begin{aligned} f(x,y) &= 3(1-x)^2 e^{-x^2-(y+1)^2} - 1/3 e^{-(x+1)^2-y^2} \\ &\quad - 10(x/5 - x^3 - y^5) e^{-x^2-y^2} \end{aligned}$$

at the points from Z, before we have computed for each of $X_{\mathrm{T}}, X_{\mathrm{E}}$ the solutions $s^*_{X_{\mathrm{T}}}, s^*_{X_{\mathrm{E}}} \in \mathcal{S}_{\phi,X}$ of (2) for all of the basis functions mentioned in the introduction. The corresponding least squares error as well as their maximal deviation can be found in Tables 1,2. The results show that for all of our examples, the choice of $X_{\mathrm{E}} \subset Z$ yields compared with $X_{\mathrm{T}} \subset Z$ better results according to the quality of the corresponding least squares approximation.

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Table 1: Results for X_{T} generated by Thinning

RBF	$\|T_Z(f - s^*_{X_{\mathrm{T}}})\|_2$	$\|T_Z(f - s^*_{X_{\mathrm{T}}})\|_\infty$
ϕ_{TPS}	0.449001690463	0.101122316498
ϕ_{MQ}	0.006064678693	0.001135391274
ϕ_{G}	0.000011040937	0.000001035253
ϕ_{IMQ}	0.012676689549	0.002504130154
ϕ_{W}	0.185800194718	0.041432420257

Table 2: Results for X_{E} output by Exchange

RBF	$\|T_Z(f - s^*_{X_{\mathrm{E}}})\|_2$	$\|T_Z(f - s^*_{X_{\mathrm{E}}})\|_\infty$
ϕ_{TPS}	0.331908066999	0.050640133774
ϕ_{MQ}	0.002792506022	0.000423751994
ϕ_{G}	0.000005503954	0.000000553759
ϕ_{IMQ}	0.005894270853	0.000987354548
ϕ_{W}	0.129807442935	0.021984218729