A priori estimates for optimal Dirichlet boundary control problems

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We consider variational discretization of control constrained elliptic Dirichlet boundary control problems on smooth two- and three-dimensional domains, where we take into account the domain approximation. The state is discretized by linear finite elements, while the control variable is not discretized. We obtain optimal error bounds for the optimal control in two and three space dimensions. Furthermore we prove a superconvergence result in two space dimensions under the assumption that the underlying finite element meshes satisfy certain regularity requirements. We confirm our findings by a numerical experiment.

1 The optimal control problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$. Given $f \in L^2(\Omega), u \in L^2(\Gamma)$ we consider the boundary value problem

$$-\Delta y + cy = f \text{ in } \Omega, \quad y = u \text{ on } \Gamma, \tag{1}$$

where $c \geq 0$ is a smooth function on $\bar{\Omega}$. It is well–known that (1) has a unique solution $y \in H^{\frac{1}{2}}(\Omega)$ in a suitable weak sense which we denote by $y = \mathcal{G}(u)$.

Next, let $\alpha > 0$ and $y_0 \in W^{1,\bar{r}}(\Omega)$, $\bar{r} > d$ be given. We then consider the Dirichlet boundary control problem

$$\min_{u \in U_{\text{ad}}} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Gamma |u|^2 \quad \text{subject to} \quad y = \mathcal{G}(u), \tag{2}$$

where $U_{\text{ad}} = \{u \in L^2(\Gamma) \,|\, a \leq u \leq b \text{ a.e. on } \Gamma\}$ and $a, b \in \mathbb{R}, a < b$. Existence of a unique solution $u \in U_{\text{ad}}$ of (2) follows by standard arguments. This solution is characterized by the variational inequality

$$\int_\Omega (y - y_0)(\mathcal{G}(v) - y) + \alpha \int_\Gamma u(v - u) \geq 0 \quad \forall v \in U_{\text{ad}}. \tag{3}$$

Let us introduce the adjoint state $p \in H^2(\Omega) \cap H^1_0(\Omega)$ as the solution of the boundary value problem

$$-\Delta p + cp = y - y_0 \text{ in } \Omega, \quad p = 0 \text{ on } \Gamma.$$

The optimal control u is then given by $u = P_{[a,b]}\left(\frac{1}{\alpha}\partial_\nu p\right)$ a.e. on Γ where $P_{[a,b]}$ denotes the pointwise projection onto the interval $[a, b]$. In [3] we prove

Lemma 1.1 *Let $u \in U_{ad}$ be the solution of (2) with corresponding state y and adjoint state p. Then*

$$u \in H^1(\Gamma), \quad y \in H^{\frac{3}{2}}(\Omega), \quad p \in W^{3,r}(\Omega) \quad \textit{for some } d < r \leq \bar{r}.$$

2 A priori error analysis

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of a polygonal domain Ω_h with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$ approximating Ω. To compare the discrete and continuous solutions we have to introduce a homeomorphism $G_h : \Omega_h \to \Omega$ with its restriction $g_h := G_{h|\Gamma_h}$ to the boundary $\Gamma_h := \partial\Omega_h$, see [3]. Let us further define the space of piecewise linear finite elements $X_h := \{v_h \in C^0(\bar{\Omega}) \,|\, v_{h|T} \in \mathcal{P}^1(T) \,\forall T \in \mathcal{T}_h\}$ and let γX_h be the restriction to Γ_h of functions in X_h.

Now we approximate (2) using the variational discretization from [4]. This leads to the following control problem depending on h:

$$\min_{u_h \in U_{h,\text{ad}}} J_h(u_h) = \frac{1}{2}\int_{\Omega_h} |y_h - y_{h,0}|^2 + \frac{\alpha}{2}\int_{\Gamma_h} |u_h|^2 \quad \text{subject to} \quad y_h = \mathcal{G}_h(u_h), \tag{4}$$

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where $U_{h,\mathrm{ad}} = \{u_h \in L^2(\Gamma_h) \,|\, a \le u_h \le b$ a.e. on $\Gamma_h\}$, $y_{h,0} = y_0 \circ G_h$ and the discrete state $y_h = \mathcal{G}_h(u_h)$ is defined by

$$\int_{\Omega_h} \nabla y_h \cdot \nabla \phi_h + c y_h \phi_h = \int_{\Omega_h} f \phi_h \; \forall \phi_h \in X_h \cap H_0^1(\Omega_h), \quad y_h = P_h(u_h) \text{ on } \Gamma_h,$$

where $P_h : L^2(\Gamma_h) \to \gamma X_h$ denotes the L^2–projection. It is not difficult to see that (4) has a unique solution $u_h \in U_{h,\mathrm{ad}}$ which is characterized by a variational inequality similar to that in (3). The following theorem is proved in [3]:

Theorem 2.1 *Let (u, y) and (u_h, y_h) be the solutions of (2) and (4). With $\tilde{u}_h = u_h \circ g_h^{-1}$ there holds*

$$\|u - \tilde{u}_h\|_{0,\Gamma} + \|y - \tilde{y}_h\|_{0,\Omega} \le Ch\sqrt{|\log h|} \quad \textit{for all } 0 < h \le h_0.$$

Remark 2.2 This result also extends to more general elliptic, coercive operators $\mathcal{A}$, see [3] .

Superconvergence. From here onwards we assume $c = 0$, $d = 2$, and that $\Omega \subset \mathbb{R}^2$ is convex.

The proof of the next theorem uses superconvergence results developed in [1] for finite element approximations to elliptic problems on (piecewise) $O(h^{2\sigma})$ irregular meshes $\mathcal{T}_h$. In [3] we prove

Theorem 2.3 *Suppose that the triangulation $\mathcal{T}_h$ is piecewise $O(h^2)$ irregular. Let (u, y) and (u_h, y_h) be the solutions of (2) and (4) (with $y_{h,0} = y_{0|\Omega_h}$). With $\tilde{u}_h = u_h \circ g_h^{-1}$ there holds*

$$\|u - \tilde{u}_h\|_{0,\Gamma} + \|y - \tilde{y}_h\|_{0,\Omega} \le Ch^{\frac{3}{2}} \quad \textit{for all } 0 < h \le h_0.$$

3 Numerical example

Let us now consider the variational discretization (4) of problem (2) with the unit circle $\Omega = B_1(0)$ as domain. We set $\alpha = 1$, $a = 0$, $b = 1$ and choose $\hat{u}(1, \phi) = \hat{u}(\phi) = \max(0, \cos\phi)$ to solve (2) with associated state $\hat{y}(r, \phi) = r^3 \max(0, \cos\phi)$ and adjoint variable $\hat{p}(r, \phi) = r^3(r - 1)\cos\phi$, where $\hat{g}(r, \phi) := g(r\cos\phi, r\sin\phi)$.

We investigate the experimental orders of convergence (EOCs) for the error functionals $E_u(h) = \|u - \tilde{u}_h\|_{0,\Gamma}$ and $E_y(h) = \|y - y_h\|_{0,\Omega_h}$, both on arbitrary, quasi-uniform meshes (see Fig. 1) and on congruently refined, piecewise $O(h^2)$ irregular meshes (see Fig. 2) for several refinement levels RL. For arbitrary meshes E_u converges linearly (Tab. 1), and with rate 1.5 on piecewise $O(h^2)$ irregular meshes (Tab. 2). In both cases the behaviour of E_y is better than predicted by Theorems 2.1, 2.3, but superconvergence on piecewise $O(h^2)$ irregular meshes is also observed.

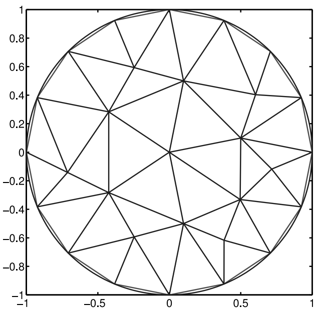

Fig. 1 Arbitrary mesh, RL = 2.

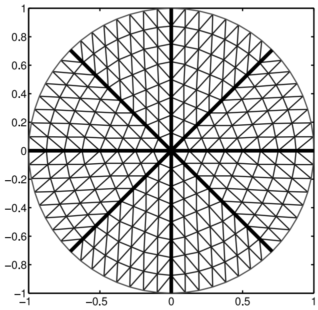

Fig. 2 Piecewise $O(h^2)$ irregular mesh, RL = 4.

Table 1 EOCs for arbitrary meshes.

RL	E_u	E_y
1-2	1.483	2.198
2-3	0.485	0.991
3-4	2.338	2.458
4-5	0.253	0.904
5-6	0.932	1.498
6-7	0.789	1.256
7-8	0.985	1.421
8-9	1.028	1.469

Table 2 EOCs for piecewise $O(h^2)$ irregular meshes.

RL	E_u	E_y
1-2	0.795	0.488
2-3	0.845	1.571
3-4	1.134	1.692
4-5	1.287	1.755
5-6	1.377	1.813
6-7	1.430	1.853
7-8	1.460	1.877
8-9	1.478	1.895

Further results related to our research can be found in [3], see also [2, 5, 6].

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