Mathematics of PDE constrained optimization
Discrete concepts
3. Constraints, interplay of discretisation and relaxation

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Topic

Optimal control of pdes with pointwise constraints:

$$\min_{u \in U_{\text{ad}}, y \in Y_{\text{ad}}} J(y, u) \text{ s.t. } PDE(y) = B(u)$$

Analysis: Casas 85,93 (pointwise state constraints), Casas & Fernandez 93 (pointwise constraints on gradient)

Numerical analysis (pointwise state constraints):

A priori:

Original problem: Casas & Mateos; Deckelnick & H.; Meyer;....
Ralaxation: Group of Rösch; Group of Tröltzsch; Hintermüller & H.; H. & Meyer; H. & Schiela; ...

A posteriori: Benedix, Vexler & Wollner; Günther & H.; Hintermüller, Hoppe & Kieweg.

Numerical analysis (pointwise constraints on gradient): Deckelnick, Günther, & H.

State and control constraints

Model problem

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2}\int_D |y - z|^2 + \frac{\alpha}{2}\|u\|_U^2$$
$$\text{subject to } y = \mathcal{G}(Bu) \text{ and } y \leq b \text{ in } D.$$

Here, $U_{ad} \subseteq U$ closed and convex, $\alpha > 0$, z, b, sufficiently smooth, and $y = \mathcal{G}(Bu)$ iff

$$Ay = Bu \text{ in } D, \text{ plus b.c. (plus i.c.)}$$

- elliptic case: $D = \Omega$ and $Ay := -\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^{d} b_i y_{x_i} + cy$ uniformly elliptic operator,
- parabolic case: $D = (0, T] \times \Omega$ and $Ay := y_t - \sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^{d} b_i y_{x_i} + cy$ with strongly elliptic leading part.

Slater condition: $\exists \tilde{u} \in U_{ad}$ such that $\mathcal{G}(B\tilde{u}) < b$ in $\bar{D}$.

Optimality conditions (Casas 86,93)

Let $u \in U_{ad}$ denote the unique optimal control and $y = \mathcal{G}(Bu)$. Then there exist $\mu \in \mathcal{M}(\bar{D})$ and some p such that there holds

$$\int_D pAv = \int_D (y-z)v + \int_{\bar{D}} v d\mu \qquad \forall v \in X,$$
$$\langle B^* p + \alpha u, v - u \rangle_{U^*,U} \geq 0 \qquad \forall v \in U_{ad},$$
$$\mu \geq 0, \ \ y \leq b \text{ in } D \text{ and } \int_{\bar{D}} (b-y) d\mu = 0,$$

where

- elliptic case: $p \in W^{1,s}(\Omega)$ for all $s < d/(d-1)$ and $X = H^2(\Omega)$ with $\sum_{i,j=1}^{d} a_{ij} v_{x_i} \nu_j = 0$ on $\partial\Omega$,
- parabolic case: $p \in L^s(W^{1,\sigma})$ for all $s, \sigma \in [1,2)$ with $2/s + d/\sigma > d+1$ and $X = \{v \in C^0(\bar{Q}); v(0,\cdot) = 0\} \cap \{v \in L^2(H^2), v_t \in L^2(H^1)\}$.

Discretization – a variational concept

Discrete optimal control problem:

$$\min_{u \in U_{ad}} J_h(u) := \frac{1}{2}\int_D |y_h - z|^2 + \frac{\alpha}{2}\|u\|_U^2$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(Bu) \text{ and } y_h \leq I_h b.$$

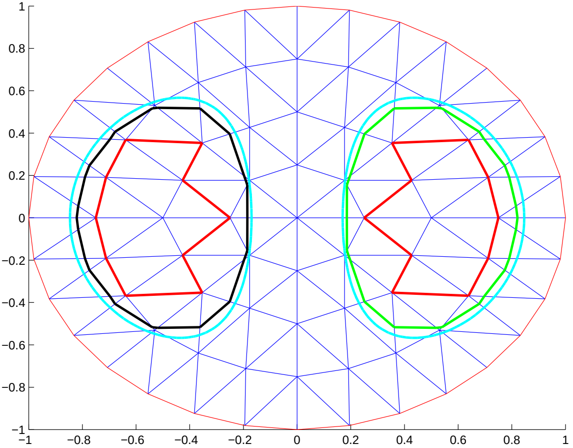
Here, $y_h(u) = \mathcal{G}_h(Bu)$ denotes the

- p.l. and continuous fe approximation to $y(u)$ (elliptic case),
- $dg(0)$ in time and p.l. and continuous fe in space approximation to $y(u)$ (parabolic case), i.e.

$$a(y_h, v_h) = \langle Bu, v_h \rangle \text{ for all } v_h \in X_h.$$

We do not discretize the control!

Variational versus conventional discretization



Variational discretization for time-dependent problems

Movie time-dependent problems

Discrete optimality conditions

Let $u_h \in U_{ad}$ denote the unique variational–discrete optimal control, $y_h = \mathcal{G}(Bu_h)$. There exist $\mu \in \mathbb{R}^k$ and $p_h \in X_h$ such that with

- $\mu_h = \sum_{j=1}^{nv} \mu_j \delta_{x_j}$ **(elliptic case, x_i fe nodes, $k = nv$),**
- $\mu_h = \sum_{i=1}^{m} \sum_{j=1}^{nv} \mu_{ij} \delta_{x_j} \circ \frac{1}{|I_i|} \int_{I_i} \bullet dt$ **(parabolic case, x_i fe nodes, I_i dg intervals, $k = nv + m$),**

we have

$$
\begin{aligned}
a(v_h, p_h) = \int_D (y_h - z) v_h + \int_{\bar{D}} v_h \, d\mu_h \qquad & \forall v_h \in X_h, \\
\langle B^* p_h + \alpha u_h, v - u_h \rangle_{U^*,U} \geq 0 \qquad & \forall v \in U_{ad}, \\
\mu_j \geq 0,\ y_h \leq I_h b, \text{ and } \int_{\bar{D}} (I_h b - y_h) \, d\mu_h = 0. &
\end{aligned}
$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

Results

Let $u_h \in U_{ad}$ denote the variational–discrete optimal solution with corresponding state $y_h \in X_h$ and $\mu_h \in \mathcal{M}(\bar{D})$. Then for h small enough

$$\|y_h\|,\ \|u_h\|_U,\ \|\mu_h\|_{\mathcal{M}(\bar{D})} \le C.$$

For the proof a discrete counterpart to the Slater condition is needed, which is deduced from uniform convergence of the discrete states associated to the Slater point $B\tilde{u}$.

Results, cont.

Let u denote the solution of the continuous problem and u_h the variational discrete optimal control. Then

$$
\begin{aligned}
&\alpha\|u-u_h\|^2+\|y-y_h\|^2 \le \\
&\quad \le C(\|\mu\|_{\mathcal{M}(\bar{D})},\|\mu_h\|_{\mathcal{M}(\bar{D})})\left\{\|y-y_h(u)\|_\infty+\|y^h(u_h)-y_h\|_\infty\right\}+ \\
&\qquad + C(\|u\|,\|u_h\|)\left\{\|y-y_h(u)\|+\|y^h(u_h)-y_h\|\right\}.
\end{aligned}
$$

Here, $y_h(u)=\mathcal{G}_h(Bu), y^h(u_h)=\mathcal{G}(Bu_h)$.

We need uniform estimates for discrete approximations.

Error estimates, parabolic case

Deckelnick, H. (JCM 2010)

Controls $u \in L^2(0,T)^m$, and $f_i \in H^1(\Omega)$ given actuations.

$$Bu := \sum_{i=1}^m u_i(t) f_i(x), \quad y_0 \in H^2(\Omega).$$

Then $y = \mathcal{G}(Bu) \in \{v \in L^\infty(H^2), v_t \in L^2(H^1)\}$ and we have with $y_h = \mathcal{G}_h(Bu)$ and time stepping $\delta t \sim h^2$

$$\|y - y_h\|_\infty \le C \begin{cases} h\sqrt{|\log h|}, & (d=2) \\ \sqrt{h}, & (d=3) \end{cases}$$

This is not an *off-the-shelf* result! It yields

$$\alpha\|u - u_h\|^2 + \|y - y_h\|^2 \le C \begin{cases} h\sqrt{|\log h|}, & (d=2) \\ \sqrt{h}, & (d=3). \end{cases}$$

Error estimates, elliptic case

Deckelnick, H. (SINUM 2007, ENUMATH 2007)

- $Bu \in L^2(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} = \begin{cases} O(h^{\frac{1}{2}}), & \text{if } d = 2, \\ O(h^{\frac{1}{4}}), & \text{if } d = 3, \end{cases}$$

- $Bu \in W^{1,s}(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \le Ch^{\frac{3}{2} - \frac{d}{2s}} \sqrt{|\log h|}.$$

- $Bu \in L^\infty(\Omega)$:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \le Ch|\log h|.$$

- $U = L^2(\Omega)$, $U_{ad} = \{u \le d\}$, u_h p.c.:

$$\|u - u_h\|_U,\ \|y - y_h\|_{H^1} \le C \begin{cases} h|\log h|, & \text{if } d = 2, \\ \sqrt{h}, & \text{if } d = 3. \end{cases}$$

Similar results obtained by C. Meyer for discrete controls.

Numerical experiment 1

$\Omega := B_1(0)$, $\alpha > 0$,

$$z(x) := 4 + \frac{1}{\pi} - \frac{1}{4\pi}|x|^2 + \frac{1}{2\pi}\log|x|,\ b(x) := |x|^2 + 4,$$

and $u_0(x) := 4 + \frac{1}{4\alpha\pi}|x|^2 - \frac{1}{2\alpha\pi}\log|x|$.

$$J(u) := \frac{1}{2}\int_\Omega |y - z|^2 + \frac{\alpha}{2}\int_\Omega |u - u_0|^2,$$

where $y = \mathcal{G}(u)$.
Unique solution $u \equiv 4$ with corresponding state $y \equiv 4$ and multipliers

$$p(x) = \frac{1}{4\pi}|x|^2 - \frac{1}{2\pi}\log|x| \quad \text{and} \quad \mu = \delta_0.$$

Experimental order of convergence

RL	$\|u - u_h\|$	$\|y - y_h\|$
1	0.788985	0.536461
2	0.759556	1.147861
3	0.919917	1.389378
4	0.966078	1.518381
5	0.986686	1.598421

Relaxing constraints – Lavrentiev (H., Meyer COAP 2008)

Lavrentiev Regularization: relax $y \le b$ to $\lambda u + y \le b$ ($\lambda > 0$). Numerical analysis yields

- $Bu^\lambda \in L^2(\Omega)$ **uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{1-d/4},$$

- $Bu^\lambda \in W^{1,s}(\Omega)$ **uniformly for all** $s \in (1, \frac{d}{d-1})$**:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h^{2-d/2-\epsilon},$$

- $Bu^\lambda \in L^\infty(\Omega), Bu_h^\lambda \in L^\infty(\Omega)$ **uniformly:**

$$\|u - u_h^\lambda\| \sim \|u - u^\lambda\| + \|u^\lambda - u_h^\lambda\| \sim \sqrt{\lambda} + h|\log h|.$$

Relaxing constraints – penalization (Hintermüller, H.)

Relax $y \le b$ with $\frac{\gamma}{2}\int_\Omega |(y-b)^+|^2 dx$ in cost functional.

- $Bu^\gamma \in L^2(\Omega)$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{1-d/4},$$

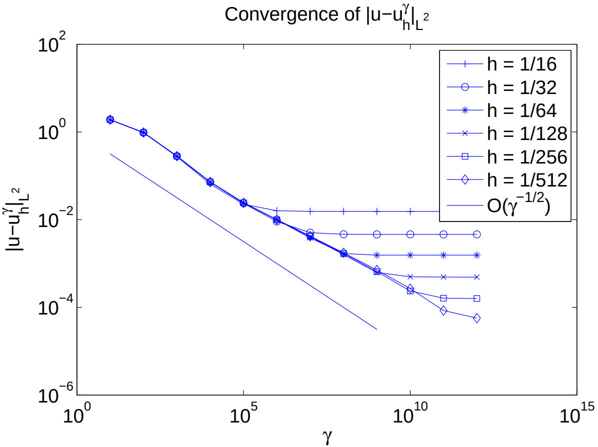
- $Bu^\gamma \in W^{1,s}(\Omega)$ **for all** $s \in (1, \frac{d}{d-1})$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h^{2-d/2-\epsilon},$$

- $Bu^\gamma \in L^\infty(\Omega), Bu_h^\gamma \in L^\infty(\Omega)$ **uniformly:**

$$\|u - u_h^\gamma\| \sim \|u - u^\gamma\| + \|u^\gamma - u_h^\gamma\| \sim$$
$$\sim \left(h^{1-d/p} + \frac{1}{\sqrt{\gamma}} h^{-d/2}\right)^{1/2} + h|\log h|.$$

Relaxing constraints – penalization, numerical results



Relaxing constraints – barriers

Barriers: relax $y \le b$ by adding $-\mu \int_\Omega \log(b-y)dx$ to cost functional ($\mu > 0$). Numerical analysis yields

- **$Bu^\mu \in L^2(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{1-d/4},$$

- **$Bu^\mu \in W^{1,s}(\Omega)$ for all $s \in (1, \frac{d}{d-1})$ uniformly:**

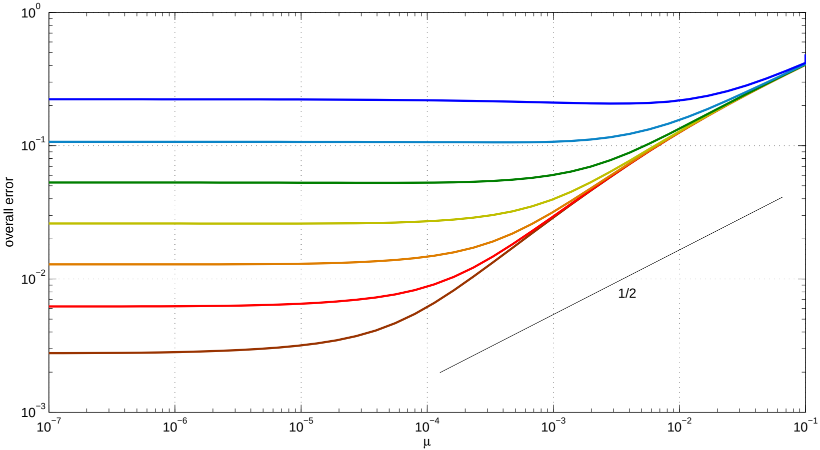
$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h^{2-d/2-\epsilon},$$

- **$Bu^\mu \in L^\infty(\Omega)$, $Bu_h^\mu \in L^\infty(\Omega)$ uniformly:**

$$\|u - u_h^\mu\| \sim \|u - u^\mu\| + \|u^\mu - u_h^\mu\| \sim \sqrt{\mu} + h|\log h|.$$

This is work in progress with Anton Schiela.

Relaxing constraints – barriers, numerical results



Consequence: Grid size h and parameters (λ, γ, μ) should be coupled;

Lavrentiev: $\sqrt{\lambda} \sim h^{2-d/2}$,

Barriers: $\sqrt{\mu} \sim h^{2-d/2}$,

Penalization ($p = \infty$): $\frac{1}{\sqrt{\gamma}} \sim h^{1+d/2}$ **(optimal ?).**

Constraints on the gradient

Consider

$$\min_{u\in U_{\mathrm{ad}}} J(u) = \frac{1}{2}\int_\Omega |y-z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+\frac{\alpha}{2}\int_\Omega |u|^2\right)$$

where $y = \mathcal{G}(u)$, **i.e. solves the pde, and** $\nabla y \in Y_{\mathrm{ad}}$.

Here

$$Y_{\mathrm{ad}} = \{z \in C^0(\bar{\Omega})^d \,|\, |z(x)| \le \delta, x \in \bar{\Omega}\},$$

and

$$\begin{aligned} r = 2: &\qquad U_{\mathrm{ad}} = \{u \in L^2(\Omega) \,|\, a \le u \le b \text{ a.e. in } \Omega\}(a, b \in L^\infty), \\ r > d: &\qquad U_{\mathrm{ad}} = L^r(\Omega). \end{aligned}$$

Then $U_{\mathrm{ad}} \subset L^r(\Omega)$ **for** $r > d \Rightarrow \nabla y \in C^0(\bar{\Omega})^d$.

Slater condition:

$$\exists \hat{u} \in U_{\mathrm{ad}} \,|\nabla \hat{y}(x)| < \delta,\ x \in \bar{\Omega},\ \text{where } \hat{y} \text{ solves the pde with } u = \hat{u}.$$

Optimality conditions (Casas & Fernandez)

An element $u \in U_{ad}$ is a solution if and only if there exist $\vec{\mu} \in \mathcal{M}(\bar{\Omega})^d$ and $p \in L^t(\Omega)$ ($t < \frac{d}{d-1}$) such that

$$\begin{array}{rll} \int_\Omega p\mathcal{A}z - \int_\Omega (y-z)z & = \int_{\bar{\Omega}} \nabla z \cdot d\vec{\mu} & \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega) \\ \int_{\bar{\Omega}} (z - \nabla y) \cdot d\vec{\mu} & \leq 0 & \forall z \in Y_{ad}, \\ \int_\Omega (p + \alpha u)(\tilde{u} - u) & \geq 0 & \forall \tilde{u} \in U_{ad} \text{ for } r = 2, \text{ or} \\ p + \alpha((u+)|u|^{r-2}u) & = 0 & \text{in } \Omega \text{ for } r > d. \end{array}$$

Structure of multiplier: $\vec{\mu} = \frac{1}{\delta} \nabla y\, \mu$, where $\mu \in \mathcal{M}(\bar{\Omega}) \geq 0$ is concentrated on $\{x \in \bar{\Omega} \,|\, |\nabla y(x)| = \delta\}$.

FE discretization, conventional

Piecewise linear, continuous Ansatz for the state $y_h = \mathcal{G}_h(u) \in X_h$.

The discrete control problem reads

$$\min_{u \in U_{\text{ad}}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \left(+\frac{\alpha}{2}\int_\Omega |u|^2\right)$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T \nabla y_h\right)_{T \in \mathcal{T}_h} \in Y^h_{\text{ad}},$$

where

$$Y^h_{\text{ad}} := \{\mathrm{c_h} : \bar\Omega \to \mathbb{R}^d \mid \mathrm{c_h}_{|T} \text{ is constant and } |\mathrm{c_h}_{|T}| \le \delta,\ T \in \mathcal{T}_h\}.$$

FE discretization, conventional, optimality conditions

The variational discrete problem has a unique solution $u_h \in U_{\text{ad}}$. There exist $\mu_T \in \mathbb{R}^d$, $T \in \mathcal{T}_{h,X}$ and $p_h \in X_h$ such that with $y_h = \mathcal{G}_h(u_h)$ we have

$$a(v_h, p_h) = \int_\Omega (y_h - z)v_h + \sum_{T \in \mathcal{T}_{h,X}} |T| \, \nabla v_{h|T} \cdot \mu_T \qquad \forall v_h \in X_h,$$

$$\sum_{T \in \mathcal{T}_{h,X}} |T| \big(c_{h_T} - \nabla y_{h|T} \big) \cdot \mu_T \le 0 \quad \forall c_h \in C_h,$$

$$p_h + \alpha((u_h+)|u_h|^{r-2} u_h) = 0 \quad \text{in } \Omega.$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \, \frac{1}{\delta} \nabla y_{h_T}$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \ge 0$ and $\mu_T > 0$ only if $|\nabla y_{h_T}| = \delta$.

Results

Deckelnick, Günther, H. (Oberwolfach Report 2008): Let $u_h \in U_{\mathrm{ad}}$ be the variational discrete optimal solution with corresponding state $y_h \in X_h$ and adjoint variables $p_h \in X_h$, $\vec{\mu}_T (T \in \mathcal{T}_h)$.

Then for h small enough

- $\|y_h\|$, $\|u_h\|_{L^r}$, $\|p_h\|_{L^{\frac{r}{r-1}}}$, $\sum_{T \in \mathcal{T}_{h,X}} |T|\,|\mu_T| \leq C$,
- $\|y - y_h\| \leq Ch^{\frac{1}{2}(1-\frac{d}{r})}$, $\|u - u_h\|_{L^r} \leq Ch^{\frac{1}{r}(1-\frac{d}{r})}$, and $\|u - u_h\|_{L^2} \leq Ch^{\frac{1}{2}(1-\frac{d}{r})}$.

These results are also valid for a piecewise constant Ansatz of the control.

FE discretization, Raviart Thomas

Mixed fe approximation of the state with lowest order Raviart–Thomas element, i.e.

$$(y_h, \mathbf{v_h}) = \mathcal{G}_h(u) \in Y_h \times \mathbf{V_h}$$

denotes the solution of

$$\begin{aligned}
\int_\Omega A^{-1}\mathbf{v_h}\cdot\mathbf{w_h} + \int_\Omega y_h\,\mathrm{div}\mathbf{w_h} &= 0 \qquad \forall \mathbf{w_h} \in \mathbf{V_h}\\
\int_\Omega z_h\,\mathrm{div}\mathbf{v_h} - \int_\Omega a_0 y_h\, z_h + \int_\Omega u\, z_h &= 0 \qquad \forall z_h \in Y_h.
\end{aligned}$$

FE discretization, cont.

The discrete control problem reads

$$\min_{u\in U_{\text{ad}}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - z|^2 + \frac{\alpha}{2}\int_\Omega |u|^2$$

$$\text{subject to } \; (y_h, \mathrm{v_h}) = \mathcal{G}_h(u) \text{ and } \left(\frac{1}{|T|}\int_T A^{-1}\mathrm{v_h}\right)_{T\in\mathcal{T}_h} \in Y_{\text{ad}}^h,$$

where

$$Y_{\text{ad}}^h := \{\mathrm{c_h} : \bar\Omega \to \mathbb{R}^d \mid \mathrm{c_h}_{|T} \text{ is constant and } |\mathrm{c_h}_{|T}| \le \delta,\ T \in \mathcal{T}_h\}.$$

FE discretization, optimality conditions

The discrete problem has a unique solution $u_h \in U_{ad}$. Furthermore, there are $\vec{\mu}_T \in \mathbb{R}^d$ and $(p_h, \chi_h) \in Y_h \times V_h$ such that with $(y_h, v_h) = \mathcal{G}_h(u_h)$ we have

$$
\begin{aligned}
\int_\Omega A^{-1}\chi_h \cdot w_h + \int_\Omega p_h \,\mathrm{div} w_h + \sum_{T\in\mathcal{T}_h} \vec{\mu}_T \cdot ⨍_T A^{-1} w_h &= 0 \,\forall w_h \in V_h \\
\int_\Omega z_h \,\mathrm{div}\chi_h - \int_\Omega a_0 p_h\, z_h + \int_\Omega (y_h - z)\, z_h &= 0 \,\forall z_h \in Y_h. \\
\int_\Omega (p_h + \alpha u_h)(\tilde{u} - u_h) &\geq 0 \,\forall \tilde{u} \in U_{ad} \\
\sum_{T\in\mathcal{T}_h} \vec{\mu}_T \cdot \left(c_{h|T} - ⨍_T A^{-1} v_h\right) &\leq 0 \,\forall c_h \in Y_{ad}^h.
\end{aligned}
$$

Structure of the multiplier: $\vec{\mu}_T = \mu_T \frac{1}{\delta} ⨍_T A^{-1} v_h$, where $\mu_T \in \mathbb{R}$. Furthermore, $\mu_T \geq 0$ and $\mu_T > 0$ only if $\left| ⨍_T A^{-1} v_h \right| = \delta$.

Results

Deckelnick, Günther, H. (Numer. Math 2008): Let $u_h \in U_{\mathrm{ad}}$ be the optimal solution of the discrete problem with corresponding state $(y_h, \mathrm{v_h}) \in Y_h \times \mathrm{V_h}$ and adjoint variables $(p_h, \chi_\mathrm{h}) \in Y_h \times \mathrm{V_h}$, $\vec{\mu}_T$, $T \in \mathcal{T}_h$.

Then for h small enough

- $\|y_h\|, \sum_{T\in\mathcal{T}_h} |\vec{\mu}_T| \le C$, **and**
- $\|u - u_h\| + \|y - y_h\| \le Ch^{\frac{1}{2}} |\log h|^{\frac{1}{2}}$.

Constraints on the gradient, example

We take $\Omega = B_2(0)$ and consider

$$\min J(u) = \frac{1}{2}\|y - z\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)}$$

with pointwise bounds on the constraints, i.e. $\{a \le u \le b\}$, where $a, b \in L^\infty(\Omega)$, and pointwise bounds on the gradient, i.e. $|\nabla y(x)| \le \delta := 1/2$. State and control satisfy

$$-\Delta y = f + u \text{ in } \Omega, \quad y = 0 \text{ on } \partial\Omega.$$

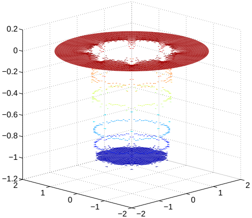
Data:

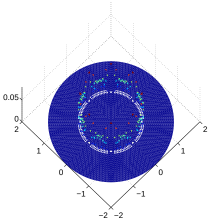
$$z(x) := \begin{cases} \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4}|x|^2 & , 0 \le |x| \le 1 \\ \frac{1}{2}\ln 2 - \frac{1}{2}\ln|x| & , 1 < |x| \le 2 \end{cases} \qquad f(x) := \begin{cases} 2 & , 0 \le |x| \le 1 \\ 0 & , 1 < |x| \le 2 \end{cases}$$

Solution:

$$y(x) \equiv z(x) \text{ and } u(x) = \begin{cases} -1 & , 0 \le |x| \le 1 \\ 0 & , 1 < |x| \le 2 \end{cases}$$

Numerical experiment, piecewise constant control Ansatz



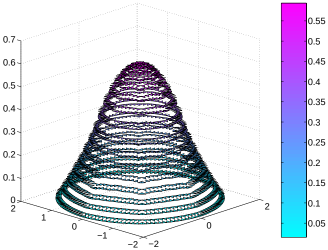


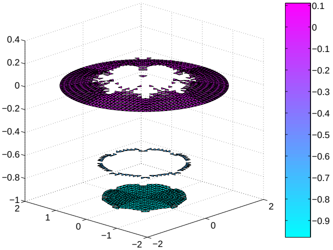
Experimental order of convergence

RL	$\|u - u_h\|_{L^4}$	$\|u - u_h\|$	$\|y - y_h\|$
1	0.76678	0.72339	1.90217
2	0.33044	0.64248	1.25741
3	0.27542	0.54054	1.23233
4	0.28570	0.53442	1.16576

Results show the predicted behaviour, since $r = \infty$.

Numerical solution, mixed finite elements





Experimental order of convergence

RL	$\|u - u_h\|$	$\|y - y_h\|$	$\|y^P - y_h^P\|$
1	0.98576	1.06726	1.08949
2	0.51814	1.02547	1.09918
3	0.50034	1.01442	1.08141

Superscript P denotes post–processed piecewise linear state. It attains the same order of convergence but yields significantly smaller approximation error.

Thank you very much for your attention