Mathematics of PDE constrained optimization
Discrete concepts
2. Tailored discretization

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Structure exploiting discretization

This can be best explained in the case without control constraints, i.e. $U_{\text{ad}} \equiv U$. Then the first order necessary optimality conditions for $(\mathbb{P})$ read

$$\nabla \hat{J}(u) = \alpha u + RB^*S^*(SBu - z) \equiv \alpha u + RB^*p = 0 \text{ in } U.$$

For proceeding on the numerical level this identity clearly gives us the advice to relate to each other the discrete Ansätze for the control u and the adjoint variable p.

This remains true also in the presence of control constraints, for which this smooth operator equation has to be replaced by the nonsmooth operator equation

$$u = P_{U_{\text{ad}}}(u - \sigma(\alpha u + RB^*p)) \equiv_{\sigma=\frac{1}{\alpha}} P_{U_{\text{ad}}}\left(-\frac{1}{\alpha}RB^*p\right) \text{ in } U, \tag{0.1}$$

where $P_{U_{\text{ad}}}$ denotes the orthogonal projection in U onto the admissible set of controls.

In any case, optimal control and corresponding adjoint state are related to each other, and this should be reflected by numerical approaches to be taken for the solution of problem $(\mathbb{P})$.

Remark

Controls should be discretized conservative, i.e. according to the relation between the adjoint state and the control given by the first order optimality condition. This rule should be obeyed in both, the First discretize, then optimize, and in the First optimize, then discretize approach.

A structure exploiting discretization concept

Let us closer investigate (0.1) in terms of the simple fixpoint iteration given next.

Algorithm

- $\boldsymbol{u}$ *given*
- *do until convergence*
  $\boldsymbol{u}^+ = P_{U_{ad}}\left(-\frac{1}{\alpha}RB^*p(u)\right),\ u = u^+.$

In this algorithm $p(u)$ is obtained by first solving $y = SBu$, and then $p = S^*(SBu - z)$.

To obtain a discrete algorithm we now replace the solution operators S, S^* by their discrete counterparts S_h, S_h^* obtained by a Finite Element discretization, say. The discrete algorithm then reads

Algorithm

- u *given*
- *do until convergence*
  $u^+ = P_{U_{ad}}\left(-\frac{1}{\alpha} RB^* p_h(u)\right)$, $u = u^+$,

where $p_h(u)$ is obtained by first solving $y = S_h Bu$, and then solving $p_h = S_h^*(S_h Bu - z)$.

We note that in this algorithm the control is not discretized. Only state and adjoint state are discretized.

Two questions immediately arise.

1. Is Algorithm 3 numerically implementable?
2. Do Algorithms 2, 3 converge?

Let us first discuss question (2). Since both algorithms are fixpoint algorithms, sufficient conditions for convergence are given by the relations

$$\alpha > \|RB^*S^*SB\|_{\mathcal{L}(U)}$$

for Algorithm 2, and by

$$\alpha > \|RB^*S_h^*S_hB\|_{\mathcal{L}(U)}$$

for Algorithm 3, since $P_{U_{\mathrm{ad}}} : U \to U_{\mathrm{ad}}$ denotes the orthogonal projection which is Lipschitz continuous with Lipschitz constant $L = 1$.

Question (1) admits the answer Yes, whenever for given u it is possible to numerically evaluate the expression

$$P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}RB^*p_h(u)\right)$$

in the $i-th$ iteration of Algorithm 3 with an numerical overhead which is independent of the iteration counter of the algorithm.

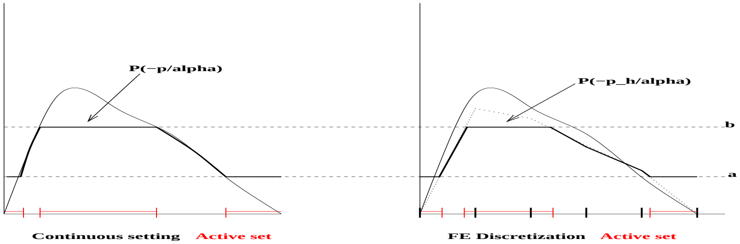
To illustrate this fact let us turn back to Example ??(1), i.e. $U = L^2(\Omega)$ and B denoting the injection, with $a \equiv const1$, $b \equiv const2$. In this case it is easy to verify that

$$P_{U_{\rm ad}}(v)(x) = P_{[a,b]}(v(x)) = \max\{a, \min\{v(x), b\}\},$$

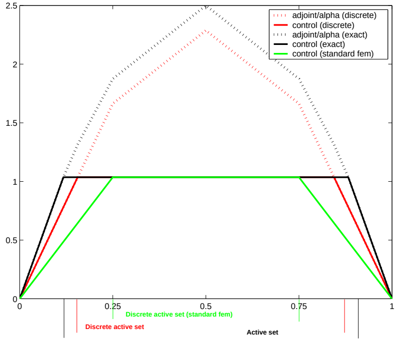
so that in every iteration of Algorithm 3 we have to form the control

$$u^+(x) = P_{[a,b]}\left(-\frac{1}{\alpha}p_h(x)\right), \tag{0.2}$$

which for in the onedimensional setting is illustrated in Figure 8.



A 1–d example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control (discrete)
adjoint/alpha (exact)
control (exact)
control (standard fem)
Discrete active set (standard fem)
Discrete active set
Active set
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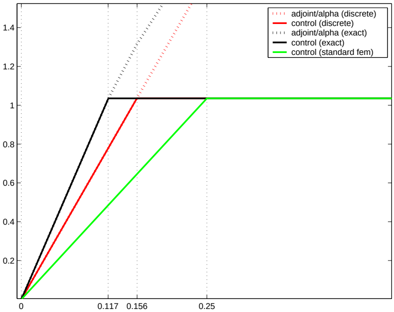
adjoint/alpha (discrete)
control (discrete)
adjoint/alpha (exact)
control (exact)
control (standard fem)
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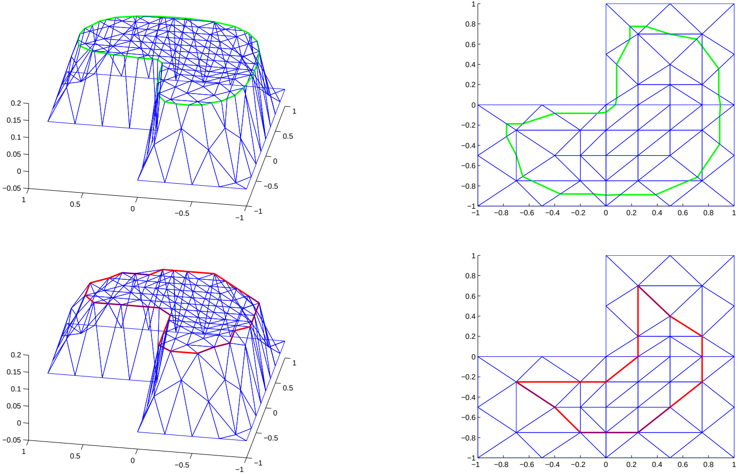
1.4

A 2–d example



To construct the function u^+ it is sufficient to characterize the intersection of the bounds a, b (understood as constant functions) and the function $-\frac{1}{\alpha}p_h$ on every simplex T of the triangulation $\tau = \tau_h$. For piecewise linear finite element approximations of p we have the following theorem.

Theorem

Let u^+ denote the function of (0.2), with p_h denoting a piecewise linear, continuous finite element function, and constant bounds $a < b$. Then there exists a partition $\kappa_h = \{K_1, \ldots K_{l(h)}\}$ of Ω such that u^+ restricted to K_j $(j = 1, \ldots, l(h))$ is a polynomial either of degree zero or one. For $l(h)$ there holds

$$l(h) \leq Cnt(h),$$

with a positive constant $C \leq 3$ and $nt(h)$ denoting the number of simplexes in τ_h. In particular, the vertices of the discrete active set associated to u^+ need not coincide with finite element nodes.

Proof: Abbreviate $\xi_h^a := -\frac{1}{\alpha}p_h^* - a$, $\xi_h^b := b - \frac{1}{\alpha}p_h^*$ and investigate the zero level sets 0_h^a and 0_h^b of ξ_h^a and ξ_h^b, respectively.

Case $n = 1$: $0_h^a \cap T_i$ is either empty or a point $S_i^a \in T_i$. Every point S_i^a subdivides T_i into two sub-intervals. Analogously $0_h^b \cap T_i$ is either empty or a point $S_i^b \in T_i$. Further $S_i^a \neq S_i^b$ since $a < b$. The maximum number of sub-intervals of T_i induced by 0_h^a and 0_h^b therefore is equal to three. Therefore, $l(h) \leq 3nt(h)$, i.e. $C = 3$.

Case $n \in \mathbb{N}$: $0_h^a \cap T_i$ is either empty or a part of a $k-$dimensional hyperplane $(k < n)$ $L_i^a \subset T_i$, analogously $0_h^b \cap T_i$ is either empty or a part of $k-$dimensional hyperplane $(k < n)$ $L_i^b \subset T_i$. Since $a < b$ the surfaces L_i^a and L_i^b do not intersect. Therefore, similar considerations as in the case $n = 1$ yield $C = 3$.

- **It is now clear that the proof of the previous theorem easily extends to functions p_h which are piecewise polynomials of degree $k \in \mathbb{N}$, and bounds a, b which are piecewise polynomials of degree $l \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively, since the difference of a, b and p_h in this case also represents a piecewise polynomial function whose projection on every element can be easily characterized.**

- **We now have that Algorithm 3 is numerically implementable, but only converges for a certain parameter range of α. A locally fast (superlinear) convergent algorithm for the numerical solution of equation (0.3) is the semi-smooth Newton method, if the function G is semi-smooth in the sense of [HIK03],[MU03, Example 5.6].**

Let us recall that (0.1) for every $\sigma > 0$ is equivalent to the equation

$$G(u) = u - P_{U_{ad}}\left(u - \sigma\nabla\hat{J}(u)\right) \equiv u - P_{U_{ad}}(u - \sigma(\alpha u + RB^*p)) \equiv$$
$$\equiv_{\sigma=\frac{1}{\alpha}} u - P_{U_{ad}}\left(-\frac{1}{\alpha}RB^*p\right) = 0 \text{ in } U, \quad (0.3)$$

so that we may apply a semi-smooth Newton algorithm, or a primal-dual active set strategy to its numerical solution.

Remark

For the choice $\sigma = \frac{1}{\alpha}$ we in certain situations obtain that the semi-smooth Newton method and the primal-dual active set strategy are equivalent, and are both numerically implementable in the discrete case.

What is the underlying discrete problem?

Let us define

$$\hat{J}_h(u) := J(S_h B u, u), \quad u \in U$$

and consider the following infinite dimensional optimization problem

$$\min_{u \in U_{\mathrm{ad}}} \hat{J}_h(u). \tag{0.4}$$

According to (??) this problem admits a unique solution $u_h \in U_{\mathrm{ad}}$ which is characterized by the variational inequality

$$(\nabla \hat{J}_h(u_h), v - u_h)_U \geq 0 \text{ forall } v \in U_{\mathrm{ad}}, \tag{0.5}$$

This variational inequality is equivalent to the non-smooth operator equation (compare (0.3))

$$G_h(u) = u - P_{U_{ad}}\left(u - \sigma\nabla\hat{J}_h(u)\right) \equiv u - P_{U_{ad}}(u - \sigma(\alpha u + RB^* p_h)) \equiv_{\sigma=\frac{1}{\alpha}}$$
$$\equiv_{\sigma=\frac{1}{\alpha}} u - P_{U_{ad}}\left(-\frac{1}{\alpha}RB^* p_h\right) = 0 \text{ in } U,$$

where similar as above

$$\nabla\hat{J}_h(u) = \alpha u + RB^* S_h^*(S_h Bu - z) \equiv \alpha u + RB^* p_h(u).$$

The considerations made above now imply that the unique solution u_h of the infinite dimensional optimization problem (0.4) can be numerically computed either by Algorithm 3 (for α large enough), or by a semi-smooth Newton method (which for $\sigma = \frac{1}{\alpha}$ coincides with the primal–dual active set strategy) (since the function G_h also is semi-smooth), however in both cases without a further discretization step.

Primal-dual active set strategy

Solve $(B \equiv Id)$

$(\alpha + S_h^* S_h)u + \hat\lambda = S_h^* z = -r$

$\Psi(u, \hat\lambda; u) := \max(\hat\lambda + \sigma(u-b), 0) + \min(\hat\lambda + \sigma(u-a), 0) = \hat\lambda$

Primal-dual active set strategy:

Initialize $u_0 = 0$, $\hat\lambda_0 = -r$; set $l = 1$, $\epsilon > 0$ small.

Loop l

$\mathcal{A}_l^a := \{\hat\lambda_{l-1} + \sigma(u_{l-1} - a) < 0\}$ $(= \{-r - S_h^* S_h u_{l-1} - \alpha a < 0\}$, if $\sigma = \alpha)$,

$\mathcal{A}_l^b := \{\hat\lambda_{l-1} + \sigma(u_{l-1} - b) > 0\}$ $(= \{-r - S_h^* S_h u_{l-1} - \alpha b > 0\}$, if $\sigma = \alpha)$,

$\mathcal{I}_l := \Omega \setminus (\mathcal{A}_l^a \cup \mathcal{A}_l^b)$.

$l \geq 2$, $\mathcal{A}_l^a = \mathcal{A}_{l-1}^a$, $\mathcal{A}_l^b = \mathcal{A}_{l-1}^b$, or $\|\Psi(u_{l-1}, \hat\lambda_{l-1}) - \hat\lambda_{l-1}\| \leq \epsilon$:

$u = u_{l-1}$, $\hat\lambda = \hat\lambda_l$, RETURN.

Otherwise

$u_l = a$ on $\mathcal{A}_l^a$, $u_l = b$ on $\mathcal{A}_l^b$, $\hat\lambda_l = 0$ on $\mathcal{I}_l$

Solve for $u_l|_{\mathcal{I}_l}$, $\hat\lambda_l|_{\mathcal{A}_l^a \cup \mathcal{A}_l^b}$

$(\alpha + S_h^* S_h)u_l + \hat\lambda_l = -r$

$l := l + 1$.

Semi-smooth Newton method

- u **given, solve until convergence**

$$G_h'(u)u^+ = -G_h(u) + G_h'(u)u, \quad u = u^+.$$

1. **This algorithm is implementable whenever the fix–point iteration is, since**

$$-G_h(u) + G_h'(u)u =$$
$$= -P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}RB^*p_h(u)\right) - \frac{1}{\alpha}P'_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}RB^*p_h(u)\right)RB^*S_h^*S_hBu.$$

2. **In certain settings (e.g. Example ??,(1)) this algorithm for every $\alpha > 0$ is locally fast convergent.**

Neumann (Robin) boundary control

$U = L^2(\Gamma)$, $Bu := \int_\Gamma u \cdot d\Gamma \in (H^1)^*(\Omega)$, $R : U^* \to U$ with $R(u, \cdot)_U = u$.

Discrete weak form

$$a(y_h, v_h) = \int_\Gamma u v_h d\Gamma \text{ for all } v_h \in W_h,$$

discrete adjoint equation

$$a(v_h, p_h) = \int_\Omega (y_h - z) v_h dx \text{ for all } v_h \in W_h.$$

Thus

$RB^* p_h = (p_h)_\Gamma$ **piecewise polynomial, continuous on the boundary grid.**

With $U_{ad} = \{a \le u \le b\}$ **we have for the variational discrete** $u_h \in U_{ad}$

$$u_h = \max\{a, \min\{-\frac{1}{\alpha}(p_h)_\Gamma, b\}\} \text{ simple cut–off at the bounds.}$$

Dirichlet boundary control

$U = L^2(\Gamma)$, $Bu := -\int_\Gamma u\partial_\eta \cdot d\Gamma \in (H_0^1(\Omega) \cap H^2(\Omega))^*$, $R : U^* \to U$ with $R(u, \cdot)_U = u$.

Discrete weak form; find $y_h \in W_h$ with

$$a(y_h, v_h) = 0 \text{ for all } v_h \in Y_h, \text{ and } y_h = \Pi(u) \in Trace(W_h)$$

discrete adjoint equation for $p_h \in Y_h$

$$a(v_h, p_h) = \int_\Omega (y_h - z)v_h dx \text{ for all } v_h \in Y_h.$$

Thus

$$u_h = P_{U_{ad}}(\frac{1}{\alpha}\kappa_h),$$

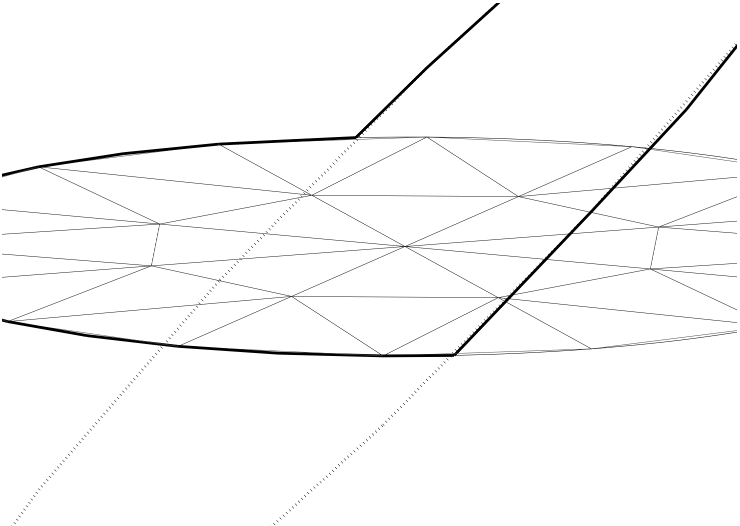
where $\kappa_h \in Trace(W_h)$ denotes the discrete adjoint flux satisfying

$$\int_\Gamma \kappa_h w_h d\Gamma = a(w_h, p_h) - \int_\Omega (y_h - z)w_h dx \text{ for all } w_h \text{ in } W_h.$$

With $U_{ad} = \{a \le u \le b\}$ we have for the variational discrete $u_h \in U_{ad}$

$$u_h = \max\{a, \min\{\frac{1}{\alpha}\kappa_h, b\}\} \text{ simple cut–off at the bounds.}$$

Dirichlet boundary control



Error estimates

Theorem

Let u denote the unique solution of (??), and u_h the unique solution of (0.4). Then there holds

$$\alpha\|u-u_h\|_U^2+\frac{1}{2}\|y(u)-y_h\|^2 \leq$$
$$\leq \langle B^*(p(u)-\tilde{p}_h(u)), u_h-u\rangle_{U*,U}+\frac{1}{2}\|y(u)-y_h(u)\|^2, \quad (0.6)$$

where $\tilde{p}_h(u) := S_h^*(SBu-z)$, $y_h(u) := S_hBu$, *and* $y(u) := SBu$.

Proof: We switch back to the variational inequalities

$$\langle \hat{J}'(u), v - u \rangle_{U^*,U} \geq 0 \text{ forall } v \in U_{\text{ad}},$$

and

$$\langle \hat{J}_h'(u_h), v - u_h \rangle_{U^*,U} \geq 0 \text{ forall } v \in U_{\text{ad}}.$$

Crucial:
The unique solution u of the continuous problem (upper inequality) is an admissible test function for the discrete problem (lower inequality).

Let us emphasize, that this is different for approaches, where the control space is discretized explictly. In this case we may only expect that u_h is an admissible test function for the continuous problem (if ever).

So let us test the optimality condition for u with u_h, and the optimality condition for u_h with u, and then add the resulting variational inequalities. This leads to

$$\langle \alpha(u-u_h) + B^*S^*(SBu - z) - B^*S_h^*(S_hBu_h - z), u_h - u\rangle_{U^*,U} \geq 0.$$

This inequality is equivalent to

$$\alpha\|u-u_h\|_U^2 \leq \langle B^*(p(u)-\tilde{p}_h(u)) + B^*(\tilde{p}_h(u) - p_h(u_h)), u_h - u)\rangle_{U^*,U}.$$

Let us investigate the second addend on the right hand side of this inequality. By definition of the adjoint variables there holds

$$\langle B^*(\tilde{p}_h(u) - p_h(u), u_h - u\rangle_{U^*,U} = \langle \tilde{p}_h(u) - p_h(u), B(u_h - u)\rangle_{Y,Y^*} =$$

$$= a(y_h - y_h(u), \tilde{p}_h(u) - p_h(u)) = \int_\Omega (y_h(u_h) - y_h(u))(y(u) - y_h(u))dx =$$

$$= -\|y_h - y\|^2 + \int_\Omega (y - y_h)(y - y_h(u))dx \leq -\frac{1}{2}\|y_h - y\|^2 + \frac{1}{2}\|y - y_h(u)\|^2$$

so that the claim of the theorem follows.

What are the consequences of this Theorem?

From the structure of this estimate we immediately infer that an error estimate for $\|u - u_h\|_U$ is at hand, if

- **an error estimate for $\|B^*(p(u) - \tilde{p}_h(u)\|_{U^*}$ is available, and**
- **an error estimate for $\|y(u) - y_h(u)\|_{L^2(\Omega)}$ is available.**

This means, that the error of $\|u - u_h\|_U$ is completely determined by the approximation properties of the discrete solution operators S_h and S_h^*.

Remark

The error $\|u - u_h\|_U$ *between the solutions* u *and* u_h *is completely determined by the approximation properties of the discrete solution operators* S_h *and* S_h^*.

Let us revisit our first example with $U = L^2(\Omega)$ and B denoting the injection. Then $y = SBu \in H^2(\Omega) \cap H_0^1(\Omega)$ (if for example $\Omega \in C^{1,1}$ or Ω polygonal, convex). Let us estimate the right side of our error estimate. There holds

$$
\begin{aligned}
(RB^*(p(u) - \tilde{p}_h(u)), u - u_h)_U = \int_\Omega (p(u) - \tilde{p}_h(u))(u - u_h)dx \leq \\
\leq \|p(u) - \tilde{p}_h(u)\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)} \leq \\
\leq ch^2 \|y(u) - z\|_{L^2(\Omega)} \|u - u_h\|_{L^2(\Omega)},
\end{aligned}
$$

and

$$\|y(u) - y_h(u)\|_{L^2} \leq ch^2 \|u\|_{L^2(\Omega)}.$$

Theorem

Let $\boldsymbol{u}$ *and* $\boldsymbol{u_h}$ *denote the solutions of the continuous and the discrete problem, respectively in the setting of the first example,(1). Then there holds*

$$\|u - u_h\|_{L^2(\Omega)} \leq ch^2 \left\{ \|y(u) - z\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right\}.$$

And this theorem is also valid for the setting of this example,(2) if we require $F_j \in L^2(\Omega)$ $(j = 1, \ldots, m)$**. This is an easy consequence of the fact that for a function** $z \in Y$ **there holds** $B^*z \in \mathbb{R}^m$ **with** $(B^*z)_i = \langle F_i, z \rangle_{Y^*,Y}$ **for** $i = 1, \ldots, m$**.**

Theorem

Let $\boldsymbol{u}$ and $\boldsymbol{u_h}$ denote the solutions of problem (**??**) *and* (0.4)*, respectively in the setting of Example* **??***(2). Then there holds*

$$\|\boldsymbol{u} - \boldsymbol{u_h}\|_{\mathbb{R}^m} \leq \boldsymbol{ch}^2 \left\{ \|\boldsymbol{y}(\boldsymbol{u}) - \boldsymbol{z}\|_{L^2(\Omega)} + \|\boldsymbol{u}\|_{\mathbb{R}^m} \right\},$$

where the positive constant now depends on the functions $\boldsymbol{F_j}$ $(\boldsymbol{j} = 1, \ldots, \boldsymbol{m})$.

Proof:

It suffices to estimate

$$
\begin{aligned}
(RB^*(p(u)-\tilde{p}_h(u)), u-u_h)_{\mathbb{R}^m} = \\
= \sum_{j=1}^{m}\left\{\int_{\Omega} F_j(p(u)-\tilde{p}_h(u))dx(u-u_h)_j\right\} \le \\
\le \|p(u)-\tilde{p}_h(u)\|_{L^2(\Omega)}\left(\sum_{j=1}^{m}\int_{\Omega}|F_j|^2 dx\right)^{\frac{1}{2}}\|u-u_h\|_{\mathbb{R}^m} \le \\
\le ch^2\|y(u)-z\|_{L^2(\Omega)}\|u-u_h\|_{\mathbb{R}^m}.
\end{aligned}
$$

The reminder terms can be estimated as above.

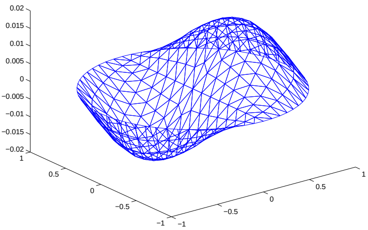
Numerical example distributed control

We consider our optimal control problem with Ω denoting the unit circle,

$$U_{\mathrm{ad}} := \{v \in L^2(\Omega); -0.2 \le u \le 0.2\} \subset L^2(\Omega)$$

and $B : L^2(\Omega) \to Y^*(\equiv H^{-1}(\Omega))$ the injection. Further we set $z(x) := (1 - |x|^2)x_1$ and $\alpha = 0.1$. The numerical discretization of state and adjoint state is performed with linear, continuous finite elements.

Here we consider the scenario that the exact solution of the problem is not known in advance (although it is easy to construct example problems where exact state, adjoint state and control are known, see [T05]). Instead we use the numerical solutions computed on a grid with $h = \frac{1}{256}$ as references.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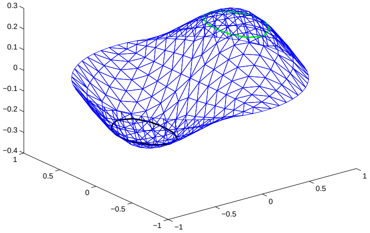
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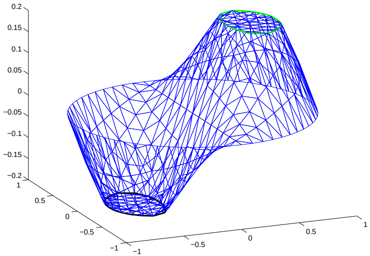
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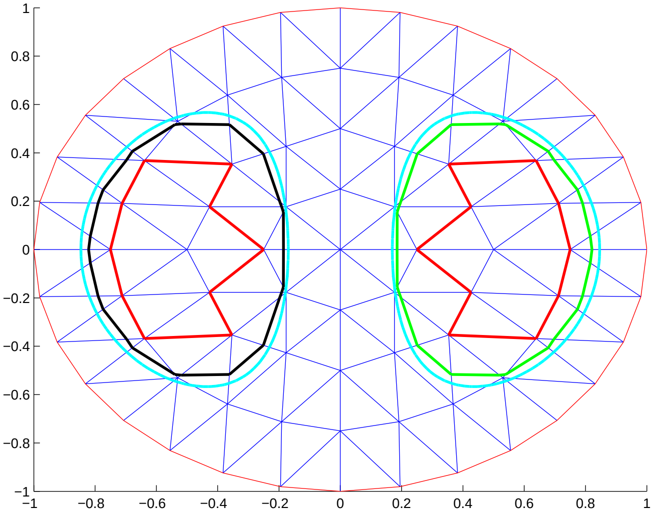
$h = \frac{1}{4}, \alpha = 0.01$



$h = \frac{1}{8}, \alpha = 0.1$

To present numerical results it is convenient to introduce the Experimental Order of Convergence, brief EOC, which for some positive error functional E is defined by

$$\mathrm{EOC} := \frac{\ln E(h_1) - \ln E(h_2)}{\ln h_1 - \ln h_2}.$$

EOC for the state y

h	$E_{y_{L2}}$	$E_{y_{sup}}$	$E_{y_{sem}}$	$E_{y_{H_1}}$	$EOC_{y_{L2}}$	$EOC_{y_{sup}}$	$EOC_{y_{H_1}}$
1/1	1.47e-2	1.63e-2	5.66e-2	5.85e-2	-	-	-
1/2	5.61e-3	6.02e-3	2.86e-2	2.92e-2	1.39	1.44	1.00
1/4	1.47e-3	1.93e-3	1.38e-2	1.39e-2	1.93	1.64	1.08
1/8	3.83e-4	5.02e-4	6.89e-3	6.90e-3	1.94	1.95	1.01
1/16	9.65e-5	1.26e-4	3.44e-3	3.45e-3	1.99	2.00	1.00
1/32	2.40e-5	3.14e-5	1.71e-3	1.71e-3	2.01	2.00	1.01
1/64	5.73e-6	7.78e-6	8.37e-4	8.37e-4	2.06	2.01	1.03
1/128	1.16e-6	1.85e-6	3.74e-4	3.74e-4	2.30	2.07	1.16

EOC for the adjoint state p

h	$E_{p_{L2}}$	$E_{p_{sup}}$	$E_{p_{sem}}$	$E_{p_{H_1}}$	$EOC_{p_{L2}}$	$EOC_{p_{sup}}$	$EOC_{p_{H_1}}$
1/1	2.33e-2	2.62e-2	8.96e-2	9.26e-2	-	-	-
1/2	6.14e-3	7.75e-3	4.36e-2	4.40e-2	1.92	1.76	1.07
1/4	1.59e-3	2.50e-3	2.17e-2	2.18e-2	1.95	1.64	1.02
1/8	4.08e-4	6.52e-4	1.09e-2	1.09e-2	1.97	1.94	0.99
1/16	1.03e-4	1.64e-4	5.48e-3	5.48e-3	1.99	1.99	1.00
1/32	2.54e-5	4.14e-5	2.73e-3	2.73e-3	2.01	1.99	1.01
1/64	6.11e-6	1.04e-5	1.33e-3	1.33e-3	2.06	1.99	1.03
1/128	1.27e-6	2.61e-6	5.96e-4	5.96e-4	2.27	1.99	1.16

EOC for the control u

h	$E_{u_{L2}}$	$E_{u_{sup}}$	$E_{u_{sem}}$	$E_{u_{H_1}}$	$EOC_{u_{L2}}$	$EOC_{u_{sup}}$	$EOC_{u_{H_1}}$
1/1	2.18e-1	2.00e-1	8.66e-1	8.93e-1	-	-	-
1/2	5.54e-2	7.75e-2	4.78e-1	4.81e-1	1.97	1.37	0.89
1/4	1.16e-2	2.30e-2	2.21e-1	2.22e-1	2.25	1.75	1.12
1/8	3.02e-3	5.79e-3	1.15e-1	1.15e-1	1.94	1.99	0.95
1/16	7.66e-4	1.47e-3	6.09e-2	6.09e-2	1.98	1.98	0.92
1/32	1.93e-4	3.67e-4	2.97e-2	2.97e-2	1.99	2.00	1.03
1/64	4.82e-5	9.38e-5	1.41e-2	1.41e-2	2.00	1.97	1.07
1/128	1.17e-5	2.37e-5	6.40e-3	6.40e-3	2.04	1.98	1.14

EOC for the control u, conventional approach

h	$E_{u_{L2}}$	$E_{u_{sup}}$	$E_{u_{sem}}$	$E_{u_{H_1}}$	$EOC_{u_{L2}}$	$EOC_{u_{sup}}$	$EOC_{u_{H_1}}$
1/1	2.18e-1	2.00e-1	8.66e-1	8.93e-1	-	-	-
1/2	6.97e-2	9.57e-2	5.10e-1	5.15e-1	1.64	1.06	0.79
1/4	1.46e-2	3.44e-2	2.39e-1	2.40e-1	2.26	1.48	1.10
1/8	4.66e-3	1.65e-2	1.53e-1	1.54e-1	1.65	1.06	0.64
1/16	1.57e-3	8.47e-3	9.94e-2	9.94e-2	1.57	0.96	0.63
1/32	5.51e-4	4.33e-3	6.70e-2	6.70e-2	1.51	0.97	0.57
1/64	1.58e-4	2.09e-3	4.05e-2	4.05e-2	1.80	1.05	0.73
1/128	4.91e-5	1.07e-3	2.50e-2	2.50e-2	1.68	0.96	0.69

$$E_a := |(A \setminus A_h) \cup (A_h \setminus A)|$$

denotes the symmetric difference of discrete and continuous active sets. EOC with the corresponding subscripts denotes the associated experimental order of convergence.

EOC for active set

	conventional	approach	our	approach
h	E_a	EOC_a	E_a	EOC_a
1/1	5.05e-1	-	5.11e-1	-
1/2	5.05e-1	0.00	3.38e-1	0.60
1/4	5.05e-1	0.00	1.25e-1	1.43
1/8	2.60e-1	0.96	2.92e-2	2.10
1/16	1.16e-1	1.16	7.30e-3	2.00
1/32	4.98e-2	1.22	1.81e-3	2.01
1/64	1.88e-2	1.41	4.08e-4	2.15
1/128	6.98e-3	1.43	8.51e-5	2.26

Postprocessing

Let us note that similar numerical results can be obtained by an approach of Meyer and Rösch presented in [MR04]. The authors in a preliminary step compute a piecewise constant optimal control $\bar{u}$ and with its help compute in a post-processing step a projected control u through

$$u = P_{U_{\text{ad}}}(-\frac{1}{\alpha}RB^* p_h(\bar{u})).$$

The numerical analysis requires the assumption, that the measure of the set of elements intersected by the boarder of the active set of the control can be bounded in terms of the grid size.

Bang–Bang control

$$\min_{u \in U_{ad}} J(u) = \frac{1}{2} \int_{\Omega} |y - y_0|^2$$
$$\text{subject to } y = \mathcal{G}(u).$$

Here,

$$U_{ad} := \{v \in L^2(\Omega); a \leq u \leq b\} \subseteq L^2(\Omega)$$

with $a < b$ constants, and $y = \mathcal{G}(Bu)$ iff

$$-\Delta y = u \text{ in } \Omega, \text{ and } y = 0 \text{ on } \partial\Omega.$$

More general elliptic operators may be considered, and also control operators which map abstract controls to feasible right-hand sides of the elliptic equation.

Existence and uniqueness, optimality conditions

The optimal control problems admits a unique solution.

The function $u \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p such that $y = \mathcal{G}(u)$, $p = \mathcal{G}(y - y_0)$ and

$$(p, v - u) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds

$$u(x) \quad \begin{cases} = a, & p(x) > 0, \\ \in [a, b], & p(x) = 0, \\ = b, & p(x) < 0. \end{cases}$$

Strict complementarity requirement for the solution u:

$$\exists C > 0 \forall \epsilon > 0 : \mathcal{L}(\{x \in \bar{\Omega}; |p(x)| \leq \epsilon\}) \leq C\epsilon$$

Variational discretization

Discrete optimal control problem:

$$\min_{u\in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(u).$$

Here, $\mathcal{G}_h(u)$ denotes the piecewise linear and continuous finite element approximation to $y(u)$, i.e.

$$a(y_h, v_h) := (\nabla y_h, \nabla v_h) = (u, v_h) \text{ for all } v_h \in X_h,$$

where on a given, quasi-uniform triangulation $\mathcal{T}_h$

$$X_h := \{w \in C^0(\bar{\Omega}); w_{|\partial\Omega} = 0, w_{|T} \text{ linear for all } T \in \mathcal{T}_h\}.$$

This problem is still $\infty-$dimensional.

Ritz projection $R_h : H_0^1(\Omega) \to X_h$,

$$a(R_h w, v_h) = a(w, v_h) \text{ for all } v_h \in X_h$$

Existence and uniqueness, optimality conditions for discrete problem

The variational-discrete optimal control problems admits a unique solution.

The function $u_h \in U_{ad}$ is a solution of the optimal control problem iff there exists an adjoint state p_h such that $y_h = \mathcal{G}_h(u_h)$, $p_h = \mathcal{G}_h(y_h - y_0)$ and

$$(p_h, v - u_h) \geq 0 \text{ for all } v \in U_{ad}.$$

There holds

$$u_h(x) \quad \begin{cases} = a, & p_h(x) > 0, \\ \in [a, b], & p_h(x) = 0, \\ = b, & p_h(x) < 0. \end{cases}$$

Error estimate

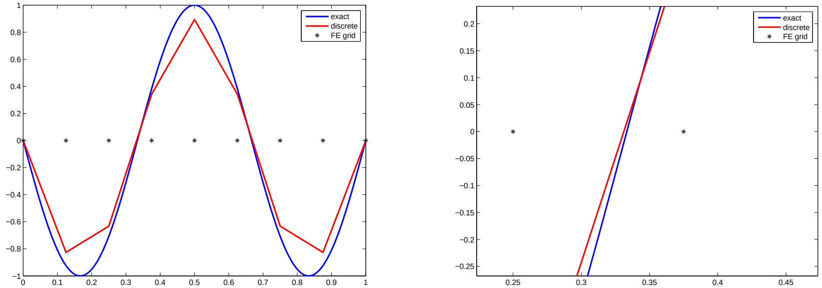
Let u, u_h denote the unique solutions of the optimal control problems with corresponding states $y = \mathcal{G}(u)$ and $y_h = \mathcal{G}_h(u_h)$, resp. Then

$$\|u - u_h\|_{L^1}, \|y - y_h\|, \|p - p_h\|_{L^\infty} \le C\left\{h^2 + \|p - R_h p\|_{L^\infty}\right\}$$

Sketch of proof:

- $\|u - u_h\|_{L^1} \le (b-a)\mathcal{L}(\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\})$
- $\{p > 0, p_h \le 0\} \cup \{p < 0, p_h \ge 0\} \subseteq \{|p(x)| \le \|p - p_h\|_\infty\} \Rightarrow$
- $\|u - u_h\|_{L^1} \le C\|p - p_h\|_\infty$
- $\|p - p_h\|_\infty \le \|p - R_h p\|_\infty + \|R_h p - p_h\|_\infty$
- $\|R_h p - p_h\|_\infty \le C\|y - y_h\|$.
- **Combine these estimates with $(p, u_h - u) \ge 0$ and $(p_h, u - u_h) \ge 0$ (note that u is admissible as testfunction for the discrete problem!).**

Numerical example with 2 switching points



Experimental order of convergence:

- **Active set 3.00073491, (here $\approx$) $\|u - u_h\|_{L^1}$: 3.00077834**
- **Function values 1.99966106**
- **$\|p - p_h\|_{L^\infty}$: 1.99979367**
- **$\|y - y_h\|_{L^\infty}$: 1.9997965**
- **$\|p - p_h\|_{L^2}$: 1.99945711**

Time–dependent problems

For the time–dependent case we sketch the analysis of Discontinuous Galerkin approximations w.r.t. time for an abstract linear–quadratic model problem. The underlying analysis turns out to be very similar to that of the previous section for the stationary model problem.

Let V, H denote separable Hilbert spaces, so that $(V, H = H^*, V^*)$ forms a Gelfand triple. We denote by $a : V \times V \to \mathbb{R}$ a bounded, coercive (and symmetric) bilinear form, and again by U the Hilbert space of controls, and by $B : U \to L^2(V^*)$ the linear control operator. Here, $T > 0$. For $y_0 \in H$ we consider the state equation

$$\left.\begin{array}{rcll} \int_0^T \langle y_t, v\rangle_{V^*,V} + a(y,v)dt & = & \int_0^T \langle (Bu)(t), v\rangle_{V^*,V} dt & \forall\, v \in L^2(V), \\ (y(0), v)_H & = & (y_0, v)_H & \forall\, v \in V, \end{array}\right\} :\Leftrightarrow y = \mathcal{T}Bu,$$

which for every $u \in U$ admits a unique solution $y = y(u) \in W := \{w \in L^2(V), w_t \in L^2(V^*)\}$.

Optimization problem

$$(TP) \quad \begin{cases} \min_{(y,u)\in W\times U_{ad}} J(y,u) := \frac{1}{2}\|y-z\|^2_{L^2(H)} + \frac{\alpha}{2}\|u\|^2_U \\ \text{s.t. } y = \mathcal{T}Bu, \end{cases} \tag{0.7}$$

where $U_{ad} \subseteq U$ denotes a closed, convex subset. Introducing the reduced cost functional

$$\hat{J}(u) := J(y(u), u),$$

the necessary (and in the present case also sufficient) optimality conditions take the form

$$\langle \hat{J}'(u), v - u\rangle_{U^*,U} \geq 0 \text{ for all } v \in U_{ad}.$$

Here

$$\nabla\hat{J}(u) = \alpha u + B^* p(y(u)),$$

where the adjoint state p solves the adjoint equation

$$\begin{aligned} \int_0^T \langle -p_t, w\rangle_{V^*,V} + a(w,p)dt &= \int_0^T (y-z,w)_H \quad && \forall\, w \in W, \\ (p(T), v)_H &= 0, && v \in V. \end{aligned}$$

This variational inequality is equivalent to the semi–smooth operator equation

$$u = P_{U_{ad}}\left(-\frac{1}{\alpha}RB^* p(y(u))\right).$$

Discretization

Let $V_h \subset V$ denote a finite dimensional subspace, and let $0 = t_0 < t_1 < \cdots < t_m = T$ denote a time grid with grid width δt. We set $I_n := (t_{n-1}, t_n]$ for $n = 1, \ldots, m$ and seek discrete states in the space

$$V_{h,\delta t} := \{\phi : [0,T] \times \Omega \to \mathbb{R}, \phi(t,\cdot)_{|\bar{\Omega}} \in V_h, \phi(\cdot, x)_{|I_n} \in \mathbb{P}_r \text{ for } n = 1, \ldots, m\}.$$

i.e. $y_{h,\delta t}$ is a polynomial of degree $r \in \mathbb{N}$ w.r.t. time. Possible choices of V_h in applications include polynomial finite element spaces, and also wavelet spaces, say. We define the discontinuous Galerkin w.r.t. time approximation (dG(r)-approximation) $\tilde{y} = y_{h,\delta t}(u) \equiv \mathcal{T}_{h,\delta t} Bu \in V_{h,\delta t}$ of the state y as unique solution of

$$A(\tilde{y}, v) := \sum_{n=1}^{m} \int_{I_n} (\tilde{y}_t, v)_H + a(\tilde{y}, v)dt + \sum_{n=1}^{m} ([\tilde{y}]^{n-1}, v^{n-1+})_H + (\tilde{y}^{0+}, v^{0+})_H =$$

$$= (y_0, v^{0+})_H + \int_0^T \langle (Bu)(t), v \rangle_{V^*,V} dt \text{ for all } v \in V_{h,\delta t}. \quad (0.8)$$

Here,

$$v^{n+} := \lim_{t \searrow t_n} v(t,\cdot), \;\; v^{n-} := \lim_{t \nearrow t_n} v(t,\cdot), \text{ and } [v]^n := v^{n+} - v^{n-}.$$

Discrete optimal control problem

The discrete counterpart of the optimal control problem reads for the variational approach

$$(P_{h,\delta t}) \qquad \min_{u \in U_{\mathrm{ad}}} \hat{J}_{h,\delta t}(u) := J(y_{h,\delta t}(u), u)$$

and it admits a unique solution $u_{h,\delta t} \in U_{\mathrm{ad}}$. We further have

$$\nabla \hat{J}_{h,\delta t}(v) = \alpha v + B^* p_{h,\delta t}(y_{h,\delta t}(v)),$$

where $p_{h,\delta t}(y_{h,\delta t}(v)) \in V_{h,\delta t}$ denotes the unique solution of

$$A(v, p_{h,\delta t}) = \int_0^T (y_{h,\delta t} - z, v)_H dt \text{ for all } v \in V_{h,\delta t}.$$

Further, the unique discrete solution $u_{h,\delta t}$ satisfies

$$\langle u_{h,\delta t} + B^* p_{h,\delta t}, v - u_{h,\delta t} \rangle_{U^*,U} \geq 0 \text{ for all } v \in U_{\mathrm{ad}}.$$

As in the continuous case this variational inequality is equivalent to a semi–smooth operator equation, namely

$$u_{h,\delta t} = P_{U_{\mathrm{ad}}} \left(-\frac{1}{\alpha} R B^* p_{h,\delta t}(y_{h,\delta t}(u_{h,\delta t})) \right).$$

Error estimate

Theorem

Let $u, u_{h,\delta t}$ denote the unique solutions of (P) and $(P_{h,\delta t})$, respectively. Then

$$\alpha\|u - u_{h,\delta t}\|_U^2 + \|y_{h,\delta t}(u_{h,\delta t}) - y_{h,\delta t}(u)\|_{L^2(H)}^2 \le$$
$$\le \langle B^*(p(u) - \tilde{p}_{h,\delta t}(u)), u_{h,\delta t} - u\rangle_{U^*,U} + \|y(u) - y_{h,\delta t}(u)\|_{L^2(H)}^2, \quad (0.9)$$

where $\tilde{p}_{h,\delta t}(u) := \mathcal{T}_{h,\delta t}^*(\mathcal{T}Bu - z)$, $y_{h,\delta t}(u) := \mathcal{T}_{h,\delta t}Bu$, *and* $y(u) := \mathcal{T}Bu$.

As a result of estimate (0.9) we have that error estimates for the variational discretization are available if error estimates for the $dg(r)$-approximation to the state and the adjoint state are available. Using the setting for the heat equation investigated by Meidner and Vexler we recover with the help of [?, Prop. 4.3,4.4] their result of [?, Corollary 5.9] for variational discretization obtained with $dG(0)$ in time and piecewise linear and continuous finite elements in space, namely

$$\alpha\|u - u_{h,\delta t}\|_U^2 + \|y_{h,\delta t}(u_{h,\delta t}) - y_{h,\delta t}(u)\|_{L^2(H)}^2 \le C\{\delta t + h^2\}.$$

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