Simulation and optimal control of the Cahn-Hilliard equation (with transport)

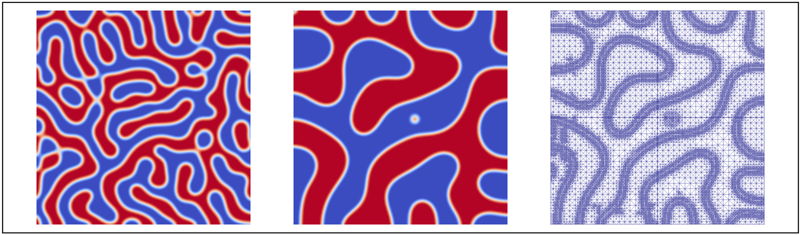
Given a binary fluid with components c_A and c_B in $\Omega \subset \mathbb{R}^d$ we simulate their evolution over a time interval I using the model of Cahn and Hilliard with double-obstacle energy and a level-set-function $c = \frac{c_A - c_B}{c_A + c_B}$:

$$(\partial_t c, v) + \frac{1}{Pe}(\nabla w, \nabla v) - (c, Eu \cdot \nabla v) = 0 \, \forall \, v \in H^1(\Omega), \, |v| \leq 1 \tag{1}$$

$$\gamma^2 (\nabla c, \nabla v - \nabla c) - (w, v - c) \geq (c, v - c) \, \forall \, v \in H^1(\Omega), \, |v| \leq 1 \tag{2}$$

$$|c| \leq 1 \tag{3}$$

for some transport $Eu : \Omega \times I \to \mathbb{R}^d$, $Eu(x,t) = \sum_{s=1}^{nu} f_s(x) u_s(t)$ and $\mathrm{div}(f_s) = 0$ and $\eta_\Omega \cdot f_s = 0$.
For the simulation an adaptive concept based on an a posteriori estimator is used.



Optimization problem:

The current aim is to use the above model as a constraint in an optimal control problem and to use the temporal part of the transport as a control yielding the following optimization problem:

$$\min J(c,u) = \frac{1}{2}\int_I \int_\Omega |c - c_d|^2 \, dxdt + \frac{\alpha}{2}\int_I |u|^2 \, dt$$

s.t. (1)–(3)

First order optimality condition (strong formulation with penalization $\phi(c)$ for $|c| \leq 1$):

$$\begin{aligned} c_t - \frac{1}{Pe}\Delta(-\gamma^2\Delta c + \phi(c) - c) + \nabla c \cdot Eu &= 0 \\ -p_t + \frac{\gamma^2}{Pe}\Delta^2 p + \frac{1}{Pe}\Delta p - \nabla p \cdot Eu - \frac{1}{Pe}\phi'(c)\Delta p - c + c_d &= 0 \\ \alpha u + E^*(c\nabla p) &= 0 \end{aligned}$$